Pairs of Languages Closed under Shuffle Projection

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Abstract. Shuffle projection is motivated by the verification of safety properties of special parameterized systems. Basic definitions and properties, especially related to alphabetic homomorphisms, are presented. The relation between iterated shuffle products and shuffle projections is shown. A special class of multi-counter automata is introduced, to formulate shuffle projection in terms of computations of these automata represented by transductions. This reformulation of shuffle projection leads to construction principles for pairs of languages closed under shuffle projection. Additionally, it is shown that under certain conditions these transductions are rational, which implies decidability of closure against shuffle projection. Decidability of these conditions is proven for regular languages. Finally, without additional conditions, decidability of the question, whether a pair of regular languages is closed under shuffle projection, is shown. In an appendix the relation between shuffle projection and the shuffle product of two languages is discussed. Additionally, a kind of shuffle product for computations in S-automata is defined.

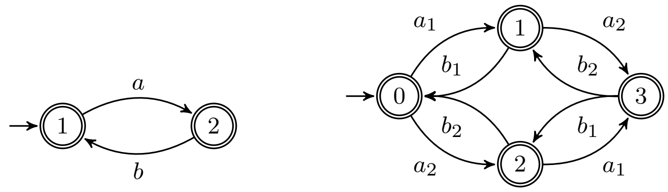
Keywords: abstractions of parameterised systems, self-similarity of system behaviour, iterated shuffle products, multicounter automata, shuffled runs of computations in multicounter automata, rational transductions, decidability of shuffle projection, simulation by Petri net

1 Introduction and Motivation

The definition of shuffle projection is motivated by our investigations of self-similarity of scalable systems [11]. Let us consider some examples:

Example 1. A server answers requests of a family of clients. The actions of the server are considered in the following. We assume w.r.t. each client that a request will be answered before a new request from this client is accepted. If the family of clients consists of only one client, then the automaton in Fig. 1(a) describes the system behavior $S \subset \Sigma^*$, where $\Sigma = \{a, b\}$, the label a depicts the *request*, and b depicts the *response*.

Example 2. Fig. 1(b) now describes the system behavior $S_{\{1,2\}} \subset \Sigma^*_{\{1,2\}}$ for two clients 1 and 2, under the assumption that the server handles the requests of different clients non-restricted concurrently.



(a) Actions at a server w.r.t. a client

(b) Two clients served concurrently by one server

Fig. 1. Scalable client-server system

For $\emptyset \neq I$ and $i \in I$ let $\Sigma_{\{i\}}$ denote pairwise disjoint copies of Σ. The elements of $\Sigma_{\{i\}}$ are denoted by a_i and $\Sigma_I := \dot{\bigcup\limits_{i \in I}} \Sigma_{\{i\}}$. Additionally let $\Sigma_\emptyset := \emptyset$, and $\Sigma_\emptyset^* := \{\varepsilon\}$. The index i describes the bijection $a \leftrightarrow a_i$ for $a \in \Sigma$ and $a_i \in \Sigma_{\{i\}}$.

Example 3. For $\emptyset \neq I \subset \mathbb{N}$ with finite I, let now $S_I \subset \Sigma_I^*$ denote the system behavior w.r.t. the client set I. For each $i \in \mathbb{N}$ $S_{\{i\}}$ is isomorphic to S, and S_I consists of the non-restricted concurrent run of all $S_{\{i\}}$ with $i \in I$.
Let $\mathcal{I}_1$ denote the set of all finite non-empty subsets of $\mathbb{N}$ (the set of all possible clients). Then, the family $(S_I)_{I \in \mathcal{I}_1}$ has the following properties:

- $I \subset K$ implies $S_I \subset S_K$ (monotony)
- $I \approx K$ implies $S_I \approx S_K$ (uniform parameterization)

Such families are called *scalable systems* [11].
Here $\approx$ denotes isomorphic. Notice, each bijection $\iota : I \to K$ defines an isomorphism $\iota_K^I : \Sigma_I^* \to \Sigma_K^*$.

In section 2 the basic definitions and properties, especially related to alphabetic homomorphisms, are presented. Section 3 shows the relations between iterated shuffle products and shuffle projections. In section 4 a special class of multi-counter automata are introduced, to formulate in section 5 shuffle projection in terms of computations of these automata. This reformulation of shuffle projection leads in section 6 to construction principles for pairs of languages closed under shuffle projection. In section 7 the results of section 5 are represented by transductions. Additionally, it is shown that under certain conditions these transductions are rational, which imply decidability of closure against shuffle projection. In section 8 decidability of these conditions is proven for regular languages. Finally, without the restrictions of section 7, decidability of the question, whether a pair of regular languages is closed under shuffle projection, is shown. In an appendix the relation between shuffle projection and the shuffle product of two languages is discussed. Additionally a kind of shuffle product for computations in S-automata is defined, which shows the results of section 5 from another point of view.

2 Basic Definitions and Homomorphic Properties

Definition 1.
For $I \subset N$ and $n \in N$ let $\tau_n^I : \Sigma_I^* \to \Sigma^*$ be the homomorphisms defined by

$$\tau_n^I(a_i) = \begin{cases} a \mid a_i \in \Sigma_{I \cap \{n\}} \\ \varepsilon \mid a_i \in \Sigma_{I \setminus \{n\}} \end{cases} .$$

For a singleton set {n}, $\tau_n^{\{n\}} : \Sigma_{\{n\}}^* \to \Sigma^*$ is an isomorphism.
For $I \in \mathcal{I}_1$ holds

$$S_I = \bigcap_{n \in I} (\tau_n^I)^{-1}(S).$$

Definition 2 ($(\dot{\mathcal{L}}(L)_I)_{I \in \mathcal{I}_1}$).
Let $\emptyset \neq L \subset \Sigma^*$ be prefix closed and

$$\dot{\mathcal{L}}(L)_I := \bigcap_{i \in I} (\tau_i^I)^{-1}(L)$$

for $I \in \mathcal{I}_1$.

The systems $\dot{\mathcal{L}}(L)_I$ consist of the "non-restricted concurrent run" of all systems $(\tau_i^{\{i\}})^{-1}(L) \subset \Sigma_{\{i\}}^*$ with $i \in I$. Because $\tau_i^{\{i\}} : \Sigma_{\{i\}}^* \to \Sigma^*$ are isomorphisms, $(\tau_i^{\{i\}})^{-1}(L)$ are pairwise disjoint copies of L.

Theorem 1.
$(\dot{\mathcal{L}}(L)_I)_{I \in \mathcal{I}_1}$ is a scalable system [11].

Now we show how to construct well-behaved systems by restricting concurrency in the behaviour-family $\dot{\mathcal{L}}$. In Example 3 holds $S_I = \dot{\mathcal{L}}(S)_I$ for $I \in \mathcal{I}_1$. If, in Example 3, the server needs specific resources for the processing of a request, then - on account of restricted resources - an non-restricted concurrent processing of requests is not possible. Thus, restrictions of concurrency in terms of synchronization conditions are necessary. One possible but very strong restriction is the requirement that the server handles the requests of different clients in the same way as it handles the requests of a single client, namely, on the request follows the response and vice versa. This synchronization condition can be formalized with the help of S and the homomorphisms Θ^I.

Definition 3.
For a set I let the homomorphism

$$\Theta^I : \Sigma_I^* \to \Sigma^* \text{ be defined by } \Theta^I(a_i) := a,$$

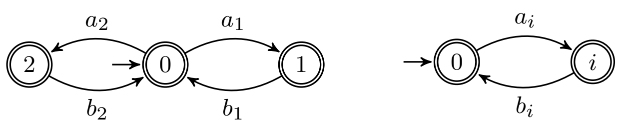
for $i \in I$ and $a \in \Sigma$.

Example 4. Restriction of concurrency on account of restricted resources: one "task" after another. All behaviors with respect to $i \in I$ *influence* each other. Let

$$\bar{S}_I := S_I \cap (\Theta^I)^{-1}(S) = \bigcap_{i \in I} (\tau_i^I)^{-1}(S) \cap (\Theta^I)^{-1}(S)$$

for $I \in \mathcal{I}_1$.

From the automaton in Fig. 1(b) it is evident that $\bar{S}_{\{1,2\}}$ will be recognized by the automaton in Fig. 2(a). Given an arbitrary $I \in \mathcal{I}_1$, then $\bar{S}_I$ is recognized



(a) Automaton recognizing $\bar{S}_{\{1,2\}}$ (b)

Fig. 2.

by an automaton with state set $\{0\} \cup I$ and state transition relation given by Fig. 2(b) for each $i \in I$. From this automaton it is evident that $(\bar{S}_I)_{I \in \mathcal{I}_1}$ is a scalable system [11].

Definition 4 ($(\bar{\mathcal{L}}(L,V)_I)_{I \in \mathcal{I}_1}$). *Let $\emptyset \neq L \subset V \subset \Sigma^*$ be prefix closed and*

$$\bar{\mathcal{L}}(L,V)_I := \bigcap_{n \in I} (\tau_n^I)^{-1}(L) \cap (\Theta^I)^{-1}(V) \text{ for } I \in \mathcal{I}_1.$$

In [11] it is shown

Theorem 2. *$(\bar{\mathcal{L}}(L,V)_I)_{I \in \mathcal{I}_1}$ is a scalable system.*

To consider arbitrary scalable systems $(\mathcal{L}_I)_{I \in \mathcal{I}}$ general parameter structures have to be defined:

Definition 5 (parameter structure). *Let N be a countable (infinite) set and $\emptyset \neq \mathcal{I} \subset 2^N \setminus \{\emptyset\}$. $\mathcal{I}$ is called a* parameter structure *based on N.*

Definition 6 (self-similar scalable system). *For arbitrary sets $I' \subset I$ let $\Pi_{I'}^I : \Sigma_I^* \to \Sigma_{I'}^*$ with*

$$\Pi_{I'}^I(a_i) = \begin{cases} a_i \mid a_i \in \Sigma_{I'} \\ \varepsilon \mid a_i \in \Sigma_I \setminus \Sigma_{I'}. \end{cases}$$

A scalable system $(\mathcal{L}_I)_{I \in \mathcal{I}}$ is called self-similar *iff*

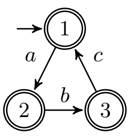
$$\Pi_{I'}^I(\mathcal{L}_I) = \mathcal{L}_{I'} \text{ for each } I, I' \in \mathcal{I} \text{ with } I' \subset I.$$

Examples: In [11] it is shown that $(S_I)_{I\in\mathcal{I}_1}$ and $(\bar{S}_I)_{I\in\mathcal{I}_1}$ are self-similar scalable systems.
In [10] it is shown that for self-similar scalable systems a large class of safety properties (uniformly parameterized safety properties) can be verified by inspecting only one corresponding "prototype system" instead of inspecting the whole family of systems. This demonstrates the importance of self-similarity for scalable systems.
The following example shows that not each $(\bar{\mathcal{L}}(L,V)_I)_{I\in\mathcal{I}_1}$ is self-similar.

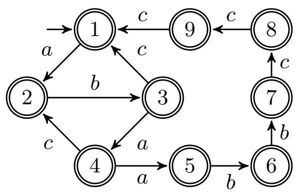
Example 5. Let $G \subset \{a,b,c\}^*$ the prefix closed language that is recognized by the automaton in Fig. 3(a). Let $H \subset \{a,b,c\}^*$ the prefix closed language that is recognized by the automaton in Fig. 3(b). It holds $\emptyset \neq G \subset H$, however, $(\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_I)_{I\in\mathcal{I}_1}$ is not self-similar, e.g.,

$$\Pi^{\{1,2,3\}}_{\{2,3\}}(\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{1,2,3\}}) \neq (\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{2,3\}} \text{ because}$$

$a_1b_1a_2a_3 \in \mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{1,2,3\}}$, and hence $a_2a_3 \in \Pi^{\{1,2,3\}}_{\{2,3\}}(\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{1,2,3\}})$, but $a_2a_3 \notin (\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{2,3\}}$.



(a) Automaton recognizing G



(b) Automaton recognizing H

Fig. 3. Counterexample

Theorem 3. *Let $\emptyset \neq L \subset V \subset \Sigma^*$ be prefix closed and $(\bar{\mathcal{L}}(L,V)_I)_{I\in\mathcal{I}_1}$ self-similar. Then*

$$\Pi^{\mathbb{N}}_K[(\bigcap_{n\in\mathbb{N}}(\tau^{\mathbb{N}}_n)^{-1}(L))\cap(\Theta^{\mathbb{N}})^{-1}(V)] \subset (\Theta^{\mathbb{N}})^{-1}(V)$$

for each subset $K \subset \mathbb{N}$.

Proof. Let $w \in \bigcap_{n\in\mathbb{N}}(\tau^{\mathbb{N}}_n)^{-1}(L)\cap(\Theta^{\mathbb{N}})^{-1}(V)$, then there exists $J \in \mathcal{I}_1$ with $w \in \Sigma^*_J$ and therefore

$$w \in \bar{\mathcal{L}}(L,V)_J. \tag{1}$$

Now

$$\Pi^{\mathbb{N}}_K(w) = \Pi^J_{K\cap J}(w). \tag{2}$$

If $K \cap J = \emptyset$, then

$$\Pi_K^{\mathbb{N}}(w) = \varepsilon \in (\Theta^{\mathbb{N}})^{-1}(V). \tag{3}$$

If $K \cap J \neq \emptyset$, then $K \cap J \in \mathcal{I}_1$. Now (1), (2) and self-similarity of $(\bar{\mathcal{L}}(L,V)_I)_{I \in \mathcal{I}_1}$ implies

$$\Pi_K^{\mathbb{N}}(w) \in \bar{\mathcal{L}}(L,V)_{K \cap J} \subset (\Theta^{K \cap J})^{-1}(V) \subset (\Theta^{\mathbb{N}})^{-1}(V). \tag{4}$$

(3) and (4) completes the proof of Theorem 3.

In [11] it is shown that

$$\Pi_K^{\mathbb{N}}[(\bigcap_{n \in \mathbb{N}} (\tau_n^{\mathbb{N}})^{-1}(L)) \cap (\Theta^{\mathbb{N}})^{-1}(V)] \subset (\Theta^{\mathbb{N}})^{-1}(V) \text{ for each subset } \emptyset \neq K \subset \mathbb{N}$$

is a sufficient condition for self-similarity of a large class of scalable systems including $(\bar{\mathcal{L}}(L,V)_I)_{I \in \mathcal{I}_1}$. So we define:

Definition 7 (closed under shuffle projection). *Let $U,V \subset \Sigma^*$. V is* closed under shuffle projection with respect to *U, iff*

$$\Pi_K^{\mathbb{N}}[(\bigcap_{n \in \mathbb{N}} (\tau_n^{\mathbb{N}})^{-1}(U)) \cap (\Theta^{\mathbb{N}})^{-1}(V)] \subset (\Theta^{\mathbb{N}})^{-1}(V) \textit{ for each subset } \emptyset \neq K \subset \mathbb{N}.$$

We abbreviate this by $\mathrm{SP}(U,V)$.

Now it holds

Corollary 1. *Let $\emptyset \neq L \subset V \subset \Sigma^*$ be prefix closed. Then $\mathrm{SP}(L,V)$ is equivalent to self-similarity of $(\bar{\mathcal{L}}(L,V)_I)_{I \in \mathcal{I}_1}$.*

Remark. It is easy to see that in Definition 7 $\mathbb{N}$ can be replaced by any set N having the same cardinality as $\mathbb{N}$ [11].

In the last section of this paper decidability of $\mathrm{SP}(U,V)$ will be proven for regular languages U and V. In preparation for this proof and supplementary to this result, first we investigate sufficient conditions for $\mathrm{SP}(U,V)$ and equivalent formulations of $\mathrm{SP}(U,V)$.

By simple set theory the definition of $\mathrm{SP}(U,V)$ has some immediate consequences:

$$\mathrm{SP}(U,V) \text{ implies } \mathrm{SP}(U',V) \text{ for each } U' \subset U. \tag{5}$$

Let $\emptyset \neq I$. Then $\mathrm{SP}(U,V_i)$ for each $i \in I$ implies

$$\mathrm{SP}(U, \bigcap_{i \in I} V_i) \text{ and } \mathrm{SP}(U, \bigcup_{i \in I} V_i). \tag{6}$$

In [11] the following theorem has been proven:

Theorem 4. *Let* $\varphi : \Sigma^* \to \Phi^*$ *be an alphabetic homomorphism and* $W, X \subset \Phi^*$, *then* $\mathrm{SP}(W, X)$ *implies* $\mathrm{SP}(\varphi^{-1}(W), \varphi^{-1}(X))$.

Because of (5) and Theorem 4

$$\mathrm{SP}(\varphi(U), V) \text{ implies } \mathrm{SP}(U, \varphi^{-1}(V)). \tag{7}$$

The inverse of implication (7) also holds. For its proof additional notations and a lemma is needed:

Let K be a non-empty set. Each alphabetic homomorphism $\varphi : \Sigma^* \to \Phi^*$ defines a homomorphism $\varphi^K : \Sigma_K^* \to \Phi_K^*$ by

$$\varphi^K(a_n) := (\varphi(a))_n \text{ for } a_n \in \Sigma_K, \text{ where } (\varepsilon)_n := \varepsilon. \tag{8}$$

If $\bar{\tau}_n^K : \Phi_K^* \to \Phi^*$ and $\bar{\Theta}^K : \Phi_K^* \to \Phi^*$ are defined analogously to τ_n^K and Θ^K, then

$$\varphi \circ \tau_n^K = \bar{\tau}_n^K \circ \varphi^K, \text{ and } \varphi \circ \Theta^K = \bar{\Theta}^K \circ \varphi^K. \tag{9}$$

Let $K \subset N$ and $\bar{\Pi}_K^N : \Phi_N^* \to \Phi_K^*$ be defined analogously to Π_K^N, then

$$\bar{\Pi}_K^N \circ \varphi^N = \varphi^K \circ \Pi_K^N. \tag{10}$$

Lemma 1. *Let* $\varphi : \Sigma^* \to \Phi^*$ *be an alphabetic homomorphism,* $U \subset \Phi^*$ *and* N *be a non-empty set, then*

$$\varphi^N\Big(\bigcap_{t \in N} (\tau_t^N)^{-1}(U)\Big) = \bigcap_{t \in N} (\bar{\tau}_t^N)^{-1}(\varphi(U)).$$

Proof. Because of (9) for $x \in \bigcap\limits_{t \in N} (\tau_t^N)^{-1}(U)$ and $t \in N$ holds

$$\bar{\tau}_t^N(\varphi^N(x)) = \varphi(\tau_t^N(x)) \in \varphi(U),$$

and therefore

$$\varphi^N\Big(\bigcap_{t \in N} (\tau_t^N)^{-1}(U)\Big) \subset \bigcap_{t \in N} (\bar{\tau}_t^N)^{-1}(\varphi(U)).$$

The contrary inclusion will be proven by the following *proposition*:

For $y \in \Phi_N^*$ let $T(y)$ be the finite set defined by $T(y) := \{t \in N \mid \bar{\tau}_t^N(y) \neq \varepsilon\}$. Then for each $y \in \Phi_N^*$ and $(u_t)_{t \in N}$ with $\bar{\tau}_t^N(y) = \varphi(u_t)$, $u_t \in \Sigma^+$ for $t \in T(y)$ and $u_t = \varepsilon$ for $t \in N \setminus T(y)$ exists an $x \in \Sigma_N^*$ with $y = \varphi^N(x)$ and $\tau_t^N(x) = u_t$ for each $t \in N$.

Proof (Proof of the proposition by induction.).
Induction base.
For $y = \varepsilon$ holds $T(y) = \emptyset$, and $x = \varepsilon$ satisfies the proposition.
Induction step.
Let $y = y'a'_s \in \Phi^*_N$ with $a'_s \in \Phi_{\{s\}}$ and $\bar{\tau}^N_t(y) = \varphi(u_t)$ with $u_t \in \Sigma^+$ for $t \in T(y)$ as well as $u_t = \varepsilon$ for $t \in N \setminus T(y)$.
Then holds $s \in T(y)$, because $\bar{\tau}^N_s(y) = \bar{\tau}^N_s(y')a'_s \neq \varepsilon$.
Let now $u_s = u'_s v'_s$ with $v'_s \in \Sigma^+$, $a'_s = \bar{\tau}^N_s(a'_s) = \varphi(v'_s) \neq \varepsilon$ and $u'_s = \varepsilon$ when $\bar{\tau}^N_s(y') = \varepsilon$.
For $t \in N \setminus \{s\}$ let $u'_t := u_t$.
$y' \in \Phi^*_N$ and $(u'_t)_{t\in N}$ now satisfy the induction hypothesis. Therefore exists $x' \in \Sigma^*_N$ with $y' = \varphi^N(x')$ and $\tau^N_t(x') = u'_t$ for each $t \in N$.
Because of the injectivity of τ^N_s on $\Sigma^*_{\{s\}}$ exists now exactly one $\tilde{v}_s \in \Sigma^+_{\{s\}}$ with $\tau^N_s(\tilde{v}_s) = v'_s$.
According to the definition of φ^N now for $\tilde{v}_s$ holds:
$\varphi^N(\tilde{v}_s) = a'_s$, hence $\varphi^N(x'\tilde{v}_s) = \varphi^N(x')\varphi^N(\tilde{v}_s) = y'a'_s = y$.
Because $\tau^N_t(x'\tilde{v}_s) = \tau^N_t(x') = u'_t = u_t$ for $t \in N \setminus \{s\}$ and $\tau^N_s(x'\tilde{v}_s) = \tau^N_s(x')\tau^N_s(\tilde{v}_s) = u'_s v'_s = u_s$ is then $x := x'\tilde{v}_s$ a proper $x \in \Sigma^*_N$ for $y = y'a'_s \in \Phi^*_N$ for the induction step. Therewith the proof of the proposition is completed.

From the above proposition follows the inclusion

$$\bigcap_{t\in N} (\bar{\tau}^N_t)^{-1}(\varphi(U)) \subset \varphi^N(\bigcap_{t\in N} (\tau^N_t)^{-1}(U)),$$

which completes the proof of Lemma 1.

Theorem 5. *Let $\varphi : \Sigma^* \to \Phi^*$ be an alphabetic homomorphism, $U \subset \Sigma^*$ and $V \subset \Phi^*$, then* $\mathrm{SP}(\varphi(U), V)$ *iff* $\mathrm{SP}(U, \varphi^{-1}(V))$.

Proof.
On account of (7) it only has to be proven that $\mathrm{SP}(U, \varphi^{-1}(V))$ implies $\mathrm{SP}(\varphi(U), V)$.

For each mapping $f : X \to Y$, $A \subset X$ and $B \subset Y$ holds

$$f(A) \cap B = f(A \cap f^{-1}(B)). \tag{11}$$

Now Lemma 1, (9) and (11) imply

$$\begin{aligned} &\bigcap_{t\in N} (\bar{\tau}^N_t)^{-1}(\varphi(u)) \cap (\bar{\Theta}^N)^{-1}(V) \\ &= \varphi^N[\bigcap_{t\in N} (\tau^N_t)^{-1}(U) \cap (\varphi^N)^{-1}((\bar{\Theta}^N)^{-1}(V))] \\ &= \varphi^N[\bigcap_{t\in N} (\tau^N_t)^{-1}(U) \cap (\Theta^N)^{-1}(\varphi^{-1}(V))] \end{aligned} \tag{12}$$

for each non-empty set N.

Because of $\varphi^K(w) = \varphi^N(w)$ for $w \in \Sigma_K^* \subset \Sigma_N^*$ and $\emptyset \neq K \subset N$, (9), (10), (12) and $\mathrm{SP}(U, \varphi^{-1}(V))$ imply

$$\begin{aligned}
&\bar{\Pi}_K^N[\bigcap_{t\in N}(\bar{\tau}_t^N)^{-1}(\varphi(u)) \cap (\bar{\Theta}^N)^{-1}(V)] \\
&= \varphi^N(\Pi_K^N[\bigcap_{t\in N}(\tau_t^N)^{-1}(U) \cap (\Theta^N)^{-1}(\varphi^{-1}(V))]) \\
&\subset \varphi^N((\Theta^N)^{-1}(\varphi^{-1}(V))) \\
&= \varphi^N((\varphi^N)^{-1}((\bar{\Theta}^N)^{-1}(V))) \subset (\bar{\Theta}^N)^{-1}(V).
\end{aligned} \tag{13}$$

(13) shows $\mathrm{SP}(\varphi(U), V)$, which completes the proof of Theorem 5.

$\mathrm{SP}(U, V)$ can be reduced to a simpler condition than Definition 7. For that purpose an additional notion and lemma is needed.

Generally for a word $w \in \Sigma_N^*$ $\kappa(w)$ denotes the smallest subset of N such that $w \in \Sigma_{\kappa(w)}^*$. More precisely

$$\kappa(\varepsilon) := \emptyset \text{ and } \kappa(wa) := \kappa(w) \cup \{i\} \text{ for } w \in \Sigma_N^* \text{ and } a \in \Sigma_{\{i\}} \text{ with } i \in N. \tag{14}$$

Lemma 2. *Let N be an infinite set, $K \subset N$ and $U \subset \Sigma^*$. Then*

$$\Pi_K^N(\bigcap_{t\in N}(\tau_t^N)^{-1}(U)) \subset \bigcap_{t\in N}(\tau_t^N)^{-1}(U).$$

Proof.
If $\varepsilon \notin U$, then $\bigcap_{t\in N}(\tau_t^N)^{-1}(U) = \emptyset$, and therefore

$$\Pi_K^N(\bigcap_{t\in N}(\tau_t^N)^{-1}(U)) = \emptyset \subset \bigcap_{t\in N}(\tau_t^N)^{-1}(U).$$

Let now $\varepsilon \in U$, and $x \in \Pi_K^N(\bigcap_{t\in N}(\tau_t^N)^{-1}(U))$, then $\tau_t^N(x) = \varepsilon \in U$ for $t \in N \setminus K$, and $\tau_t^N(x) = \tau_t^N(w) \in U$ for $w \in \bigcap_{t\in N}(\tau_t^N)^{-1}(U)$ with $\Pi_K^N(w) = x$ and $t \in K$, which implies $x \in \bigcap_{t\in N}(\tau_t^N)^{-1}(U)$. This completes the proof of the lemma.

Theorem 6. *Let $U, V \in \Sigma^*$, then $\mathrm{SP}(U, V)$, iff there exists an infinite countable set N such that*

$$\Pi_{N\setminus\{n\}}^N[(\bigcap_{t\in N}(\tau_t^N)^{-1}(U)) \cap (\Theta^N)^{-1}(V)] \subset (\Theta^N)^{-1}(V) \tag{15}$$

for each $n \in N$.

Proof. Let $K \subset N$ and $w \in \Sigma_N^*$, then by (14) holds

$$\Pi_K^N(w) = \Pi_{(N\setminus\kappa(w))\cup K}^N(w) = \Pi_{N\setminus(\kappa(w)\setminus K)}^N(w).$$

Therefore $\mathrm{SP}(U,V)$ iff there exists an infinite countable set N such that

$$\Pi_{N\setminus R}^N[(\bigcap_{t\in N}(\tau_t^N)^{-1}(U))\cap(\Theta^N)^{-1}(V)] \subset (\Theta^N)^{-1}(V) \tag{16}$$

for each finite subset $R \subset N$.

Now it is sufficient to show that (16) follows from (15).

Proof (by induction on the cardinality of $R \subset N$).
Induction base.
(16) holds for $R = \emptyset$.
Induction step.
Let $R = R' \cup \{n\}$ with $n \in N \setminus R'$, then

$$\Pi_{N\setminus R}^N = \Pi_{N\setminus R'}^{N\setminus\{n\}} \circ \Pi_{N\setminus\{n\}}^N. \tag{17}$$

On account of $\Pi_{N\setminus R'}^N(L) = \Pi_{N\setminus R'}^{N\setminus\{n\}}(L)$ for $L \subset \Sigma_{N\setminus\{n\}}^* \subset \Sigma_N^*$ (17) implies

$$\begin{aligned}&\Pi_{N\setminus R}^N[(\bigcap_{t\in N}(\tau_t^N)^{-1}(U))\cap(\Theta^N)^{-1}(V)] = \\ &\Pi_{N\setminus R'}^N[\Pi_{N\setminus\{n\}}^N[(\bigcap_{t\in N}(\tau_t^N)^{-1}(U))\cap(\Theta^N)^{-1}(V)]].\end{aligned} \tag{18}$$

By Lemma 2 holds

$$\Pi_K^N(\bigcap_{t\in N}(\tau_t^N)^{-1}(U)) \subset \bigcap_{t\in N}(\tau_t^N)^{-1}(U),$$

and therefore

$$\Pi_{N\setminus\{n\}}^N[(\bigcap_{t\in N}(\tau_t^N)^{-1}(U))\cap(\Theta^N)^{-1}(V)] \subset \bigcap_{t\in N}(\tau_t^N)^{-1}(U). \tag{19}$$

Now (15) and (19) imply

$$\Pi_{N\setminus\{n\}}^N[(\bigcap_{t\in N}(\tau_t^N)^{-1}(U))\cap(\Theta^N)^{-1}(V)] \subset (\bigcap_{t\in N}(\tau_t^N)^{-1}(U))\cap(\Theta^N)^{-1}(V). \tag{20}$$

From (18), (20) and the induction hypothesis it follows

$$\begin{aligned}&\Pi_{N\setminus R}^N[(\bigcap_{t\in N}(\tau_t^N)^{-1}(U))\cap(\Theta^N)^{-1}(V)] \subset \\ &\Pi_{N\setminus R'}^N[(\bigcap_{t\in N}(\tau_t^N)^{-1}(U))\cap(\Theta^N)^{-1}(V)] \subset (\Theta^N)^{-1}(V),\end{aligned}$$

which completes the induction step and the proof of Theorem 6.

Because of $\bigcap_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(U) = \emptyset$ for $U \subset \Sigma^+$, then trivially holds $\mathrm{SP}(U,V)$ for each $V \subset \Sigma^*$. Therefore in the following sections we consider $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ for $P, V \subset \Sigma^*$.

3 Iterated Shuffle Products

Definition 7 and the examples of scalable systems considered so far are related to iterated shuffle products.

Definition 8 (iterated shuffle product $P^{\sqcup\!\sqcup}$). *For $P \subset \Sigma^*$ let*

$P^{\sqcup\!\sqcup} := \Theta^{\mathbb{N}}[\bigcap_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})]$.
$P^{\sqcup\!\sqcup}$ *is called the* iterated shuffle product of P*.*

An immediate consequence of this definition is

$$\emptyset^{\sqcup\!\sqcup} = \{\varepsilon\}^{\sqcup\!\sqcup} = \{\varepsilon\},\ P \cup \{\varepsilon\} \subset P^{\sqcup\!\sqcup} \text{ and } P^{\sqcup\!\sqcup} \subset L^{\sqcup\!\sqcup} \text{ for } P \subset L \subset \Sigma^*. \tag{21}$$

For an alphabetic homomorphism $\varphi : \Sigma^* \to \Phi^*$ and $L \subset \Sigma^*$ holds $xy \in \varphi(L)$ iff there exist $u, v \in \Sigma^*$ with $x = \varphi(u)$, $y = \varphi(v)$ and $uv \in L$. This implies

$$\varphi(\mathrm{pre}(L)) = \mathrm{pre}(\varphi(L) \text{ for each } L \subset \Sigma^*. \tag{22}$$

where $\mathrm{pre}(M)$ denotes the set of all prefixes of words $w \in M$.

As $\Theta^{\mathbb{N}}$ and $\tau_t^{\mathbb{N}}$ are alphabetic homomorphisms, (22) implies

$$\mathrm{pre}(P^{\sqcup\!\sqcup}) = (\mathrm{pre}(P))^{\sqcup\!\sqcup}. \tag{23}$$

Example 6. Let $P = \{ab\}$, then $aabb \in P^{\sqcup\!\sqcup}$, because $aabb = \Theta^{\mathbb{N}}(a_1a_2b_2b_1)$, $\tau_1^{\mathbb{N}}(a_1a_2b_2b_1) = \tau_2^{\mathbb{N}}(a_1a_2b_2b_1) = ab \in P$ and $\tau_t^{\mathbb{N}}(a_1a_2b_2b_1) = \varepsilon$ for $t \in \mathbb{N} \setminus \{1,2\}$.

$a_1a_2b_2b_1$ is called a *structured representation* of $aabb$.
In this term $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ is a property of a certain set of structured representations, which implies

Theorem 7. *Let $P, V \subset \Sigma^*$, then $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ implies $\mathrm{SP}(P^{\sqcup\!\sqcup}, V)$.*

For the proof of Theorem 7 additional notions and three lemmas from [9] are needed. Let S and T be non-empty sets. For each $\emptyset \neq S' \subset S$ and $\emptyset \neq T' \subset T$ let

$$\Theta_{S'}^{S' \times T'} : \Sigma_{S' \times T'}^* \to \Sigma_{S'}^* \text{ with } \Theta_{S'}^{S' \times T'}(a_{(s,t)}) := a_s \text{ for each } a_{(s,t)} \in \Sigma_{S' \times T'} \text{ and}$$

$$\Theta_{T'}^{S' \times T'} : \Sigma_{S' \times T'}^* \to \Sigma_{T'}^* \text{ with } \Theta_{T'}^{S' \times T'}(a_{(s,t)}) := a_t \text{ for each } a_{(s,t)} \in \Sigma_{S' \times T'}.$$

Lemma 3 (Shuffle-lemma 1).
Let S, T be non-empty sets and $M \subset \Sigma^*$, then
$\bigcap_{s\in S}(\tau_s^S)^{-1}[\Theta^T(\bigcap_{t\in T}(\tau_t^T)^{-1}(M))] = \Theta_S^{S\times T}[\bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(M)]$,
which implies
$\Theta^S[\bigcap_{s\in S}(\tau_s^S)^{-1}[\Theta^T(\bigcap_{t\in T}(\tau_t^T)^{-1}(M))]] = \Theta^{S\times T}[\bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(M)]$,
because of
$\Theta^{S\times T} = \Theta^S \circ \Theta_S^{S\times T}$.

Lemma 4 (Shuffle-lemma 2).
Let S, T be non-empty sets and $M \subset \Sigma^*$. If a bijection between S and T exists, then $\Theta^S[\bigcap_{s\in S}(\tau_s^S)^{-1}(M)] = \Theta^T[\bigcap_{t\in T}(\tau_t^T)^{-1}(M)]$.

Definition 9 (structured representation).
Let S be a non-empty set and $M \subset \Sigma^*$. For each $x \in \Theta^S[\bigcap_{s\in S}(\tau_s^S)^{-1}(M)]$ there exists $u \in \bigcap_{s\in S}(\tau_s^S)^{-1}(M)$ such that $x = \Theta^S(u)$. We call u a structured representation of x w.r.t. S and M.

For $x \in \Sigma^*$ let $SR_M^S(x) := (\Theta^S)^{-1}(x) \cap [\bigcap_{s\in S}(\tau_s^S)^{-1}(M)]$. It is the set of all structured representations of x w.r.t. S and M.

Remark. Now $x \in P^{\sqcup\!\sqcup}$ iff there exists an infinite countable set S with $SR^S_{(P\cup\{\epsilon\})}(x) \neq \emptyset$. Therefore in Definition 8 $\mathbb{N}$ can be replaced by any infinite countable set N.

Lemma 5 (Shuffle-lemma 3).
Let S, T be non-empty sets, $M \subset \Sigma^*$, and $y \in \Sigma^*_{S\times T}$ with $\tau_{(s,t)}^{S\times T}(y) \in M$ for each $(s,t) \in S\times T$ and $x = \Theta_S^{S\times T}(y) \in \Sigma_S^*$, then
$\Pi_{S'\times T}^{S\times T}(y) \in SR_M^{S'\times T}(\Theta^{S'}(\Pi_{S'}^S(x)))$ *for each $\emptyset \neq S' \subset S$.*

Remark. The hypotheses of this lemma are given by lemma 3.

Proof (Proof of Theorem 7).

Let $x \in \bigcap_{s\in S}(\tau_s^S)^{-1}(P^{\sqcup\!\sqcup}) \cap (\Theta^S)^{-1}(V)$, where S is a countable infinite set, then $x \in \bigcap_{s\in S}(\tau_s^S)^{-1}[\Theta^T(\bigcap_{t\in T}(\tau_t^T)^{-1}(P\cup\{\varepsilon\}))]$, where T is a countable infinite set. By Lemma 3 there exists $y \in \bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(P\cup\{\varepsilon\})$ with $x = \Theta_S^{S\times T}(y)$. This implies $y \in \bigcap_{(s,t)\in S\times T}(\tau_{(s,t)}^{S\times T})^{-1}(P\cup\{\varepsilon\}) \cap (\Theta^{S\times T})^{-1}(V)$ because of $\Theta^S(x) \in V$ and $\Theta^{S\times T} = \Theta^S \circ \Theta_S^{S\times T}$. Now, by the assumption $\mathrm{SP}(P\cup\{\varepsilon\}, V)$ holds

$$\Pi_{S'\times T}^{S\times T}(y) \in (\Theta^{S\times T})^{-1}(V) \text{ for each } \emptyset \neq S' \subset S. \tag{24}$$

As now x and y fulfill the assumptions of Lemma 5, it follows

$$\Theta^{S'\times T}(\Pi^{S\times T}_{S'\times T}(y)) = \Theta^{S'}(\Pi^{S}_{S'}(x)). \tag{25}$$

Because of

$$\Theta^{S'}(\Pi^{S}_{S'}(x)) = \Theta^{S}(\Pi^{S}_{S'}(x))$$

and

$$\Theta^{S'\times T}(\Pi^{S\times T}_{S'\times T}(y)) = \Theta^{S\times T}(\Pi^{S\times T}_{S'\times T}(y))$$

(24) and (25) imply $\Pi^{S}_{S'}(x) \in (\Theta^{S})^{-1}(V)$ for each $\emptyset \neq S' \subset S$, which completes the proof of Theorem 7.

(5), (6), Theorem 4 and Theorem 7 show that in many cases it is sufficient to prove $SP(U,V)$ for very simple U. On account of our focus on system behavior, we are especially interested in $SP(U,V)$ for prefix closed languages U and V.

In Definition 8 the iterated shuffle product is represented by the homomorphic image of a set of structured representations. To get a deeper insight into the property $\mathrm{SP}(P \cup \{\varepsilon\}, V)$, in the next section we will represent $P^{\sqcup\!\sqcup}$ by an homomorphic image of a set of computations of a certain automaton. For this purpose we need a "bracketed coding" of words.

Definition 10.
Together with an alphabet Σ we consider four pairwise disjoint copies of Σ, namely $\tilde{\Sigma}$, $\mathring{\Sigma}$, $\bar{\Sigma}$, $\tilde{\bar{\Sigma}}$, and a homomorphism $\wedge : \hat{\Sigma}^* \to \Sigma^*$ with $\hat{\Sigma} := \tilde{\Sigma} \cup\!\!\!\cdot\; \mathring{\Sigma} \cup\!\!\!\cdot\; \bar{\Sigma} \cup\!\!\!\cdot\; \tilde{\bar{\Sigma}}$ and $\wedge(\tilde{a}) := \wedge(\mathring{a}) := \wedge(\bar{a}) := \wedge(\tilde{\bar{a}}) := a$ for each $a \in \Sigma$, where $\tilde{a}$, $\mathring{a}$, $\bar{a}$ and $\tilde{\bar{a}}$ are the corresponding copies of a letter $a \in \Sigma$.

For words $u \in P \subset \Sigma^+$ the four alphabets are used to characterize start-, inner-, end-, or start-end letters of u.

Definition 11.
Let $|x| \in \mathbb{N}_0$ denotes the length *of a word $x \in \Sigma^*$, defined by $|\varepsilon| := 0$ and $|xa| := |x| + 1$ for $a \in \Sigma$ and $x \in \Sigma^*$.*

The following definition depends on the fact that each $u \in \Sigma^*$ with $|u| > 1$ can be uniquely represented by $u = awb$ with $a, b \in \Sigma$ and $w \in \Sigma^*$.

Definition 12.
Let $\langle\rangle : \Sigma^* \to \{\varepsilon\} \cup \tilde{\bar{\Sigma}} \cup \tilde{\Sigma}\mathring{\Sigma}^*\bar{\Sigma}$ be the mapping defined by $\langle\rangle(\varepsilon) := \varepsilon, \langle\rangle(a) := \tilde{\bar{a}}$ for $a \in \Sigma$ and $\langle\rangle(awb) := \tilde{a}\mathring{w}\bar{b}$ for $a, b \in \Sigma$ and $w \in \Sigma^*$, where $\mathring{w}$ is defined by $\mathring{w} \in \mathring{\Sigma}^*$ and $\wedge(\mathring{w}) = w$ for each $w \in \Sigma^*$.
For short we write $\langle u\rangle$ instead of $\langle\rangle(u)$ for each $u \in \Sigma^*$.

For each $y \in \{\varepsilon\} \cup \tilde{\bar{\Sigma}} \cup \tilde{\Sigma}\mathring{\Sigma}^*\bar{\Sigma}$ holds $\langle\wedge(y)\rangle = y$, and for each $x \in \Sigma^*$ holds $\wedge(\langle x\rangle) = x$. Therefore $\langle\rangle$ is a bijection with

$$\langle\rangle^{-1} = \wedge_{|\{\varepsilon\}\cup\tilde{\bar{\Sigma}}\cup\tilde{\Sigma}\mathring{\Sigma}^*\bar{\Sigma}} \text{ and } |\langle w\rangle| = w \text{ for each } w \in \Sigma^*. \tag{26}$$

The bijection $\langle\rangle$ formalizes the "bracketed coding" of words.

By (26) and (22) holds $\wedge(\langle U\rangle) = U$ and $\wedge(\mathrm{pre}(\langle U\rangle)) = \mathrm{pre}(U)$ for each $U \subset \Sigma^*$. Therefore Theorem 5 implies

Corollary 2.
For each $U,V \subset \Sigma^*$ holds $\mathrm{SP}(U,V)$ iff $\mathrm{SP}(\langle U\rangle, \wedge^{-1}(V))$, and $\mathrm{SP}(\mathrm{pre}(U),V)$ iff $\mathrm{SP}(\mathrm{pre}(\langle U\rangle), \wedge^{-1}(V))$.

The following theorem together with its corollary prepares the automata representations of iterated shuffle products.

Theorem 8.
Let $\varphi : \Sigma^* \to \Phi^*$ be an alphabetic homomorphism and $P \subset \Sigma^*$, then holds $\varphi(P^{⧢}) = (\varphi(P))^{⧢}$.

Proof.
Let N be an infinite countable set. Let $\varphi^N : \Sigma_N^* \to \Phi_N^*$, $\bar{\tau}_n^N : \Phi_N^* \to \Phi^*$ and $\bar{\Theta}^N : \Phi_N^* \to \Phi^*$ be defined as in context of Lemma 1. Because of (9) holds

$$\mu(P^{⧢}) = \bar{\Theta}^N[\varphi^N(\bigcap_{t\in N}(\tau_t^N)^{-1}(P \cup \{\varepsilon\}))]. \tag{27}$$

From this it follows that $\mu(P^{⧢}) = (\mu(P))^{⧢}$ if the following equation holds:

$$\varphi^N(\bigcap_{t\in N}(\tau_t^N)^{-1}(P \cup \{\varepsilon\})) = \bigcap_{t\in N}(\bar{\tau}_t^N)^{-1}(\varphi(P) \cup \{\varepsilon\}) \tag{28}$$

Proof. Proof of equation (28):
Because of (9) holds

$$\bar{\tau}_t^N(\varphi^N(x)) = \varphi(\tau_t^N(x)) \in \varphi(P \cup \{\varepsilon\}) = \varphi(P) \cup \{\varepsilon\}$$

for each $x \in \bigcap_{t\in N}(\tau_t^N)^{-1}(P \cup \{\varepsilon\})$ and $t \in N$, which implies

$$\varphi^N(\bigcap_{t\in N}(\tau_t^N)^{-1}(P \cup \{\varepsilon\})) \subset \bigcap_{t\in N}(\bar{\tau}_t^N)^{-1}(\mu(P) \cup \{\varepsilon\}).$$

The other inclusion of equation (28)follows from Lemma 1, which completes the proof of equation (28) and of Theorem 8.

Because of $P = \wedge(\langle P\rangle)$ and $\mathrm{pre}(P) = \wedge(\mathrm{pre}(\langle P\rangle))$ Theorem 8 implies

Corollary 3.
Let $P \subset \Sigma^*$, then $P^{⧢} = \wedge(\langle P\rangle^{⧢})$, and $(\mathrm{pre}(P))^{⧢} = \wedge((\mathrm{pre}(\langle P\rangle))^{⧢})$.

Therefore Corollary 3 reduces automata representations of $P^{⧢}$ rsp. $(\mathrm{pre}(P))^{⧢}$ to automata representations of $\langle P\rangle^{⧢}$ rsp. $(\mathrm{pre}(\langle P\rangle))^{⧢}$.

4 Automata Representations of Iterated Shuffle Products

Automata representations of iterated shuffle products are well known. See for example [2] and [6], where multicounter automata are considered. Therefore the purpose of this section is not to introduce a new automaton concept, but to establish notions for further investigations of $\mathrm{SP}(P,V)$ based on computations of these automata. On account of Corollary 3 we start with an automaton representation for $(\mathrm{pre}(\langle P\rangle))^{\sqcup\!\sqcup}$.

Let $P \subset \Sigma^*$ and $\mathbb{P} = (\Sigma, Q, \delta, q_0, F)$ be a (not necessarily finite) deterministic automaton recognizing P, where $\delta : Q \times \Sigma \to Q$ is a partial function, $q_0 \in Q$ and $F \subset Q$. As usual, δ is extended to a partial function $\delta : Q \times \Sigma^* \to Q$. For simplicity we assume $P \neq \emptyset$ and $\delta(q_0, \mathrm{pre}(P)) = Q$.

Moreover, we take this set of conditions as a general assumption for the rest of the paper.

The idea to define a semiautomaton (automaton without final states [1]) $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$ recognizing $(\mathrm{pre}(\langle P\rangle))^{\sqcup\!\sqcup}$ is the following: Each computation in $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$ "correspond" to a "shuffled run" of several not necessarily recognizing computations in $\mathbb{P}$, which we call *"elementary computations"*. For each $q \in Q$ the states of $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$ store the number of "elementary computations" which just have reached the state q in such a "shuffled run" of "elementary computations".

Formally, the state set of $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$ is $\mathbb{N}_0^Q$, the set of all functions $f : Q \to \mathbb{N}_0$.

Let $0 \in \mathbb{N}_0^Q$ be defined by $0(q) := 0$ for each $q \in Q$. For $q \in Q$ and $k \in \mathbb{N}$ let $k_q \in \mathbb{N}_0^Q$ be defined by $k_q(x) := \begin{cases} k \mid x = q \\ 0 \mid x \in Q \setminus \{q\} \end{cases}$.

For $f, g \in \mathbb{N}_0^Q$ let

- $f \geqslant g$ iff $f(x) \geqslant g(x)$ for each $x \in Q$,
- $f + g \in \mathbb{N}_0^Q$ with $(f+g)(x) := f(x) + g(x)$ for each $x \in Q$, and
- for $f \geqslant g$, $f - g \in \mathbb{N}_0^Q$ with $(f-g)(x) := f(x) - g(x)$ for each $x \in Q$.

The state transition relation $\hat{\sqcup\!\sqcup}_{\mathbb{P}}$ of $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$ is composed of four disjunct subsets whose elements describe

- the "entry into a new elementary computation",
- the "transition within an open elementary computation",
- the "completion of an open elementary computation",
- the "entry into a new elementary computation with simultaneous completion of this elementary computation".

Definition 13 ($\hat{S}$-automaton $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$).
$\hat{\mathbb{P}}_{\sqcup\!\sqcup} = (\hat{\Sigma}, \mathbb{N}_0^Q, \hat{\sqcup\!\sqcup}_{\mathbb{P}}, 0)$ *w.r.t.* $\mathbb{P}$ *is a semiautomaton with an infinite state set* $\mathbb{N}_0^Q$,

the initial state 0 *and a state transition relation* $\hat{\sqcup}_{\mathbb{P}} \subset \mathbb{N}_0^Q \times \hat{\Sigma} \times \mathbb{N}_0^Q$ *defined by*

$$\begin{aligned}
\hat{\sqcup}_{\mathbb{P}} :=& \tilde{\sqcup}_{\mathbb{P}} \;\dot\cup\; \mathring{\sqcup}_{\mathbb{P}} \;\dot\cup\; \bar{\sqcup}_{\mathbb{P}} \;\dot\cup\; \tilde{\bar{\sqcup}}_{\mathbb{P}} \;\; \textit{with} \\
\tilde{\sqcup}_{\mathbb{P}} :=& \{(f,x,f+1_p) \in \mathbb{N}_0^Q \times \tilde{\Sigma} \times \mathbb{N}_0^Q \mid \delta(q_0, \wedge(x)) = p \textit{ and it exists } b \in \Sigma \textit{ such} \\
& \textit{that } \delta(p,b) \textit{ is defined}\}, \\
\mathring{\sqcup}_{\mathbb{P}} :=& \{(f,x,f+1_p-1_q) \in \mathbb{N}_0^Q \times \mathring{\Sigma} \times \mathbb{N}_0^Q \mid f \geqslant 1_q, \delta(q, \wedge(x)) = p \textit{ and it exists} \\
& b \in \Sigma \textit{ such that } \delta(p,b) \textit{ is defined}\}, \\
\bar{\sqcup}_{\mathbb{P}} :=& \{(f,x,f-1_q) \in \mathbb{N}_0^Q \times \bar{\Sigma} \times \mathbb{N}_0^Q \mid f \geqslant 1_q \textit{ and } \delta(q, \wedge(x)) \in F\} \textit{ and} \\
\tilde{\bar{\sqcup}}_{\mathbb{P}} :=& \{(f,x,f) \in \mathbb{N}_0^Q \times \tilde{\bar{\Sigma}} \times \mathbb{N}_0^Q \mid \delta(q_0, \wedge(x)) \in F\}.
\end{aligned}$$

Generally $\hat{\mathbb{P}}_{\sqcup}$ is an infinite nondeterministic semiautomaton.

Example 7. $P = \{abc, abbc\}$ Two computations in $\hat{\mathbb{P}}_{\sqcup}$:

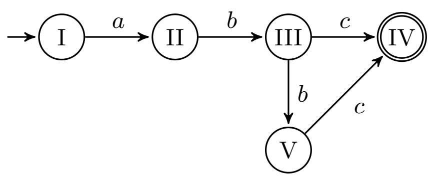


Fig. 4. Automaton $\mathbb{P}$ recognizing P

$$0 \xrightarrow{\tilde{a}} 1_{\mathrm{II}} \xrightarrow{\mathring{b}} 1_{\mathrm{III}} \xrightarrow{\tilde{a}} 1_{\mathrm{III}} + 1_{\mathrm{II}} \xrightarrow{\mathring{b}} 1_{\mathrm{V}} + 1_{\mathrm{II}} \ldots$$

$$0 \xrightarrow{\tilde{a}} 1_{\mathrm{II}} \xrightarrow{\mathring{b}} 1_{\mathrm{III}} \xrightarrow{\tilde{a}} 1_{\mathrm{III}} + 1_{\mathrm{II}} \xrightarrow{\mathring{b}} 2_{\mathrm{III}} \ldots$$

$\hat{A}_{\mathbb{P}} \subset \hat{\sqcup}_{\mathbb{P}}^*$ denotes the set of all paths in $\hat{\mathbb{P}}_{\sqcup}$ starting with the initial state 0 and including the empty path ε. For $w \in \hat{A}_{\mathbb{P}}$, $\hat{Z}_{\mathbb{P}}(w)$ denotes the final state of the path w and $\hat{Z}_{\mathbb{P}}(\varepsilon) := 0$. Formally the prefix closed language $\hat{A}_{\mathbb{P}}$ and the function $\hat{Z}_{\mathbb{P}} : \hat{A}_{\mathbb{P}} \to \mathbb{N}_0^Q$ are defined inductively by

$$\varepsilon \in \hat{A}_{\mathbb{P}},\; \hat{Z}_{\mathbb{P}}(\varepsilon) := 0,\; w(f,x,g) \in \hat{A}_{\mathbb{P}} \text{ and } \hat{Z}_{\mathbb{P}}(w(f,x,g)) := g \tag{29}$$

for $w \in \hat{A}_{\mathbb{P}}$, $\hat{Z}_{\mathbb{P}}(w) = f$ and $(f,x,g) \in \hat{\sqcup}_{\mathbb{P}}$.

Let the function $\hat{\alpha}_{\mathbb{P}} : \hat{A}_{\mathbb{P}} \to \hat{\Sigma}^*$ be inductively defined by

$$\hat{\alpha}_{\mathbb{P}}(\varepsilon) := \varepsilon \text{ and } \hat{\alpha}_{\mathbb{P}}(w(f,x,g)) := \hat{\alpha}_{\mathbb{P}}(w)x \tag{30}$$

for $w(f,x,g) \in \hat{A}_{\mathbb{P}}$ and $(f,x,g) \in \hat{\sqcup}_{\mathbb{P}}$. $\hat{\alpha}_{\mathbb{P}}(u)$ is called the *label of a path u*.

Definition 14.
Let N be an infinite countable set. For $I' \subset I \subset N$ and $t \in N$ let $\hat{\Sigma}_{\{t\}} := \tilde{\Sigma}_{\{t\}} \,\dot\cup\, \mathring{\Sigma}_{\{t\}} \,\dot\cup\, \bar{\Sigma}_{\{t\}} \,\dot\cup\, \bar{\tilde{\Sigma}}_{\{t\}}$, $\hat{\tau}^I_t : \hat{\Sigma}^*_I \to \hat{\Sigma}^*$, $\hat{\Theta}^I : \hat{\Sigma}^*_I \to \hat{\Sigma}^*$ and $\hat{\Pi}^I_{I'} : \hat{\Sigma}^*_I \to \hat{\Sigma}^*_{I'}$ be defined according to the definitions of $\hat{\Sigma}$, $\Sigma_{\{t\}}$, τ^I_t, Θ^I and $\Pi^I_{I'}$, where $\hat{\Sigma}_I := \dot{\bigcup\limits_{s\in I}} \hat{\Sigma}_{\{s\}}$.

The key to prove that $\hat{\mathbb{P}}_{\sqcup}$ recognizes $(\mathrm{pre}(\langle P\rangle))^{\sqcup} = \hat{\Theta}^N[\bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))]$ is to define an appropriate function $\hat{c}_{\mathbb{P}} : \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle)) \to \hat{A}_{\mathbb{P}}$. For that purpose we first consider the function $\hat{n}_{\mathbb{P}} : \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle)) \to \mathbb{N}^Q_0$, defined by

$$\hat{n}_{\mathbb{P}}(x)(q) := \#(\{t \in N \mid \delta(q_0, \wedge(\hat{\tau}^N_t(x))) = q \text{ and } \hat{\tau}^N_t(x) \notin \langle P\rangle \cup \{\varepsilon\}\}) \tag{31}$$

for each $x \in \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))$ and $q \in Q$, where $\#(M)$ denotes the cardinality of a set M.

As in (23) it holds

$$\mathrm{pre}(\bigcap_{t\in N} (\hat{\tau}^N_t)^{-1}(\langle P \cup \{\varepsilon\}\rangle)) = \bigcap_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle)). \tag{32}$$

This shows that $\bigcap\limits_{t\in N} (\hat{\tau}t^N)^{-1}(\mathrm{pre}(\langle P\rangle))$ is a prefix closed language.

The following property of $\hat{n}_{\mathbb{P}}$ is the key for the definition of $\hat{c}_{\mathbb{P}}$.

Lemma 6.
Let $x\hat{a} \in \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))$ with $\hat{a} \in \hat{\Sigma}_N$, then $(\hat{n}_{\mathbb{P}}(x), \hat{\Theta}^N(\hat{a}), \hat{n}_{\mathbb{P}}(x\hat{a})) \in \hat{\sqcup}_{\mathbb{P}}$.

Proof.
For $I \subset N$ an immediate consequence of Lemma 2 and the definitions of $\hat{n}_{\mathbb{P}}$ and $\hat{\sqcup}_{\mathbb{P}}$ is

$$\hat{\Pi}^N_I[\bigcap_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))] \subset \bigcap_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle)) \cap \hat{\Sigma}^*_I, \tag{33}$$

$$\hat{n}_{\mathbb{P}}(x) = \hat{n}_{\mathbb{P}}(\hat{\Pi}^N_I(x)) + \hat{n}_{\mathbb{P}}(\hat{\Pi}^N_{N\setminus I}(x)) \text{ for } x \in \bigcap_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle)) \text{ and} \tag{34}$$

$$(f,b,g) \in \hat{\sqcup}_{\mathbb{P}} \text{ implies } (f+h, b, g+h) \in \hat{\sqcup}_{\mathbb{P}} \text{ for } h \in \mathbb{N}^Q_0. \tag{35}$$

For $x\hat{a} \in \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))$ there exists $s \in N$ with $\hat{a} \in \hat{\Sigma}_{\{s\}}$, and therefore $\hat{\Pi}^N_{N\setminus\{s\}}(x\hat{a}) = \hat{\Pi}^N_{N\setminus\{s\}}(x)$. Now by (33) - (35) it is sufficient to prove the lemma for $x\hat{a} \in \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle)) \cap \hat{\Sigma}^*_{\{s\}} = (\hat{\tau}^{\{s\}}_s)^{-1}(\mathrm{pre}(\langle P\rangle))$, where $\hat{\tau}^{\{s\}}_s : \hat{\Sigma}^*_{\{s\}} \to \hat{\Sigma}^*$ is a bijection.

For $w \in (\hat{\tau}^{\{s\}}_s)^{-1}(\mathrm{pre}(\langle P\rangle) \setminus \langle P\rangle \cup \{\varepsilon\})$ holds $\hat{n}_{\mathbb{P}}(w) = 1_q$, with $\delta(q_0, \wedge(\hat{\tau}^{\{s\}}_s(w))) = q$ and for $w \in (\hat{\tau}^{\{s\}}_s)^{-1}(\langle P\rangle \cup \{\varepsilon\})$ holds $\hat{n}_{\mathbb{P}}(w) = 0$. Therefore the definition of $\hat{\sqcup}_{\mathbb{P}}$ immediately implies $(\hat{n}_{\mathbb{P}}(x), \hat{\Theta}^N(\hat{a}), \hat{n}_{\mathbb{P}}(x\hat{a})) = (\hat{n}_{\mathbb{P}}(x), \hat{\tau}^{\{s\}}_s(\hat{a}), \hat{n}_{\mathbb{P}}(x\hat{a})) \in \hat{\sqcup}_{\mathbb{P}}$ for $x\hat{a} \in (\hat{\tau}^{\{s\}}_s)^{-1}(\mathrm{pre}(\langle P\rangle))$, which completes the proof of the lemma.

Lemma 6 makes the following definition sound:

Definition 15.
Let the function $\hat{c}_{\mathbb{P}} : \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle)) \to \hat{A}_{\mathbb{P}}$ *be inductively defined by* $\hat{c}_{\mathbb{P}}(\varepsilon) := \varepsilon$ *and* $\hat{c}_{\mathbb{P}}(x\hat{a}) := \hat{c}_{\mathbb{P}}(x)(\hat{n}_{\mathbb{P}}(x), \hat{\Theta}^N(\hat{a}), \hat{n}_{\mathbb{P}}(x\hat{a}))$ *for* $x\hat{a} \in \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))$ *with* $\hat{a} \in \hat{\Sigma}_N$.

This definition immediately implies

Theorem 9. *Let* $x \in \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))$ *then*

$$\hat{Z}_{\mathbb{P}}(\hat{c}_{\mathbb{P}}(x)) = \hat{n}_{\mathbb{P}}(x), \tag{36a}$$

$$\hat{\alpha}_{\mathbb{P}}(\hat{c}_{\mathbb{P}}(x)) = \hat{\Theta}^N(x), \tag{36b}$$

$$|\hat{c}_{\mathbb{P}}(x)| = |x|, \ \textit{and} \tag{36c}$$

$$\mathrm{pre}(\hat{c}_{\mathbb{P}}(x)) = \hat{c}_{\mathbb{P}}(\mathrm{pre}(x)). \tag{36d}$$

To prove surjectivity of $\hat{c}_{\mathbb{P}}$ we need a counterpart of Lemma 6:

Lemma 7. *Let* $c(f,\hat{b},g) \in \hat{A}_{\mathbb{P}}$ *with* $(f,\hat{b},g) \in \hat{\sqcup}_{\mathbb{P}}$, *and* $w \in \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))$ *with* $\hat{c}_{\mathbb{P}}(w) = c$.

If $\hat{b} \in \tilde{\Sigma} \dot{\cup} \tilde{\tilde{\Sigma}}$, *then for each* $\hat{a} \in \hat{\Sigma}_{N\setminus\kappa(w)}$ *with* $\hat{\Theta}^N(\hat{a}) = \hat{b}$ *holds* $w\hat{a} \in \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))$ *and* $\hat{c}_{\mathbb{P}}(w\hat{a}) = c(f,\hat{b},g)$.
If $\hat{b} \in \mathring{\Sigma} \dot{\cup} \bar{\Sigma}$, *then there exists* $\hat{a} \in \hat{\Sigma}_{\kappa(w)}$ *with* $\hat{\Theta}^N(\hat{a}) = \hat{b}$ *such that* $w\hat{a} \in \bigcap\limits_{t\in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P\rangle))$ *and* $\hat{c}_{\mathbb{P}}(w\hat{a}) = c(f,\hat{b},g)$.

Proof.
By the definition of $\hat{\sqcup}_{\mathbb{P}}$ each $(f,\hat{b},g) \in \tilde{\sqcup}_{\mathbb{P}} \dot{\cup} \tilde{\tilde{\sqcup}}_{\mathbb{P}}$ can be represented by

$$(f,\hat{b},g) = (f,\hat{b},f+h) \text{ with } (0,\hat{b},h) \in \tilde{\sqcup}_{\mathbb{P}} \dot{\cup} \tilde{\tilde{\sqcup}}_{\mathbb{P}}, \tag{37}$$

and each $(f,\hat{b},g) \in \mathring{\sqcup}_{\mathbb{P}} \dot{\cup} \bar{\sqcup}_{\mathbb{P}}$ can be represented by

$$(f,\hat{b},g) = (f'+1_q,\hat{b},f'+k) \text{ with } q \in Q \text{ and } (1_q,\hat{b},k) \in \mathring{\sqcup}_{\mathbb{P}} \dot{\cup} \bar{\sqcup}_{\mathbb{P}}. \tag{38}$$

In these representations h is uniquely determined by $\hat{b}$, and k is uniquely determined by q and $\hat{b}$. More precisely: There exist partial functions $\tilde{\bar{\delta}}_{\mathbb{P}} : \tilde{\Sigma} \dot{\cup} \tilde{\bar{\Sigma}} \to \mathbb{N}_0^Q$ and $\mathring{\bar{\delta}}_{\mathbb{P}} : Q \times (\mathring{\Sigma} \dot{\cup} \bar{\Sigma}) \to \mathbb{N}_0^Q$ such that

$$(0, \hat{b}, h) \in \tilde{\sqcup}_{\mathbb{P}} \dot{\cup} \tilde{\bar{\sqcup}}_{\mathbb{P}} \text{ iff } h = \tilde{\bar{\delta}}_{\mathbb{P}}(\hat{b}), \text{ and } (1_q, \hat{b}, k) \in \mathring{\sqcup}_{\mathbb{P}} \dot{\cup} \bar{\sqcup}_{\mathbb{P}} \text{ iff } k = \mathring{\bar{\delta}}_{\mathbb{P}}(q, \hat{b}). \quad (39)$$

Let $w\hat{a} \in \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle))$ with $\hat{\Theta}^N(\hat{a}) = \hat{b} \in \tilde{\Sigma} \dot{\cup} \tilde{\bar{\Sigma}}$, then $\hat{a} \in \hat{\Sigma}_{N \setminus \kappa(w)}$. Now (34) and the definition of $\tilde{\bar{\delta}}_{\mathbb{P}}$ imply

$$\hat{n}_{\mathbb{P}}(w\hat{a}) = \hat{n}_{\mathbb{P}}(w) + \tilde{\bar{\delta}}_{\mathbb{P}}(\hat{b}). \quad (40)$$

Let $w\hat{a} \in \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle))$ with $\hat{\Theta}^N(\hat{a}) = \hat{b} \in \mathring{\Sigma} \dot{\cup} \bar{\Sigma}$, then there exists $s \in \kappa(w)$ such that $\hat{\tau}_s^N(w) \neq \varepsilon$ and $\hat{\tau}_s^N(w)\hat{b} \in \mathrm{pre}(\langle P \rangle)$. Now (34) and the definition of $\mathring{\bar{\delta}}_{\mathbb{P}}$ imply

$$\hat{n}_{\mathbb{P}}(w\hat{a}) = \hat{n}_{\mathbb{P}}(\hat{\Pi}^N_{N \setminus \{s\}}(w) + \mathring{\bar{\delta}}_{\mathbb{P}}(\delta(q_0, \wedge(\hat{\tau}_s^N(w))), \hat{b}). \quad (41)$$

Let $\hat{b} \in \tilde{\Sigma} \dot{\cup} \tilde{\bar{\Sigma}}$, then $(f, \hat{b}, g) = (f, \hat{b}, f + \tilde{\bar{\delta}}_{\mathbb{P}}(\hat{b})) \in \tilde{\sqcup}_{\mathbb{P}} \dot{\cup} \tilde{\bar{\sqcup}}_{\mathbb{P}}$ implies $\hat{b} \in \mathrm{pre}(\langle P \rangle)$, and therefore $w \in \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle))$ implies

$$w\hat{a} \in \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle)) \text{ for each } \hat{a} \in \hat{\Sigma}_{N \setminus \kappa(w)} \text{ with } \hat{\Theta}^N(\hat{a}) = \hat{b}. \quad (42)$$

Now (37), (39), (40) and (42) prove the first part of Lemma 7.

Let $\hat{b} \in \mathring{\Sigma} \dot{\cup} \bar{\Sigma}$, and let $(f, \hat{b}, g) \in \mathring{\sqcup}_{\mathbb{P}} \dot{\cup} \bar{\sqcup}_{\mathbb{P}}$ be represented by $(f, \hat{b}, g) = (f' + 1_q, \hat{b}, f' + \mathring{\bar{\delta}}_{\mathbb{P}}(q, \hat{b}))$ with $q \in Q$. On account of $f' + 1_q = \hat{n}_{\mathbb{P}}(w)$, there exists $s \in \kappa(w)$ such that $f' = \hat{n}_{\mathbb{P}}(\hat{\Pi}^N_{N \setminus \{s\}}(w))$, $\hat{\tau}_s^N(w) \notin \langle P \rangle \cup \{\varepsilon\}$, and $\delta(q_0, \wedge(\hat{\tau}_s^N(w))) = q$. Therefore by $(f' + 1_q, \hat{b}, f' + \mathring{\bar{\delta}}_{\mathbb{P}}(q, \hat{b})) \in \mathring{\sqcup}_{\mathbb{P}} \dot{\cup} \bar{\sqcup}_{\mathbb{P}}$ holds $\hat{\tau}_s^N(w)\hat{b} \in \mathrm{pre}(\langle P \rangle)$, which implies

$$w\hat{a} \in \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle)) \text{ for } \hat{a} \in \hat{\Sigma}_{\{s\}} \text{ with } \hat{\Theta}^N(\hat{a}) = \hat{b}. \quad (43)$$

Now (38), (39), (41) and (43) prove the second part of Lemma 7.

Generally, for $L \subset \Sigma^*$ and $x \in \Sigma^*$ the *left quotient* $x^{-1}(L)$ is defined by

$$x^{-1}(L) := \{y \in \Sigma^* | xy \in L\}. \quad (44)$$

By induction on the length of $c \in \hat{A}_{\mathbb{P}}$ Lemma 7 implies

Theorem 10.

$$\hat{c}_{\mathbb{P}}[\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))] = \hat{A}_{\mathbb{P}}. \quad \textit{Moreover} \tag{45a}$$

$$\hat{c}_{\mathbb{P}}[x(x^{-1}[\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))])] = \hat{c}_{\mathbb{P}}(x)[(\hat{c}_{\mathbb{P}}(x))^{-1}(\hat{A}_{\mathbb{P}})] \tag{45b}$$

for each $x \in \bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))$.

On account of (36b) now from (45a) it follows

Corollary 4.
$(\mathrm{pre}(\langle P\rangle))^{\sqcup\!\sqcup} = \hat{\alpha}_{\mathbb{P}}(\hat{A}_{\mathbb{P}})$, *which states that the semiautomaton* $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$ *recognizes the prefix closed language* $(\mathrm{pre}(\langle P\rangle))^{\sqcup\!\sqcup}$.

Because of

$$\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\langle P\rangle \cup \{\varepsilon\}) =$$
$$\{x \in \bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle)) \mid \hat{\tau}_t^N(x) \in \langle P\rangle \cup \{\varepsilon\} \text{ for each } t \in N\} =$$
$$\{x \in \bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle)) \mid \hat{n}_{\mathbb{P}}(x) = 0\},$$

it holds

$$\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\langle P\rangle \cup \{\varepsilon\}) = \hat{n}_{\mathbb{P}}^{-1}(0). \tag{46}$$

Therefore (36a) and (45a) imply

$$\hat{c}_{\mathbb{P}}[\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\langle P\rangle \cup \{\varepsilon\})] = \hat{Z}_{\mathbb{P}}^{-1}(0). \tag{47}$$

Now, from (47) and (36b) it follows

Corollary 5.
$\langle P\rangle^{\sqcup\!\sqcup} = \hat{\alpha}_{\mathbb{P}}(\hat{Z}_{\mathbb{P}}^{-1}(0))$, *which states that the semiautomaton* $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$ *enriched by the final state* $0 \in \mathbb{N}_0^Q$ *recognizes* $\langle P\rangle^{\sqcup\!\sqcup}$.

Let $\mathbb{A}$ be an automaton recognizing $L \subset \Phi^*$ and let $\varphi : \Phi^* \to \Gamma^*$ be a *strictly* alphabetic homomorphism, where strictly is defined by $|\varphi(w)| = |w|$ for each $w \in \Phi^*$. Then it is easy and well known to construct an automaton $\mathbb{A}'$ recognizing $\varphi(L) \subset \Gamma^*$. Now this construction will be realized for the semiautomaton $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$ and the strictly alphabetic homomorphism $\wedge : \hat{\Sigma}^* \to \Sigma^*$. Additionally this construction will be extended to a modification of the function $\hat{c}_{\mathbb{P}}$.

Definition 16 (S**-automaton** $\mathbb{P}_{\sqcup\!\sqcup}$**).**
$\mathbb{P}_{\sqcup\!\sqcup} = (\Sigma, \mathbb{N}_0^Q, \sqcup\!\sqcup_{\mathbb{P}}, 0)$ *w.r.t.* $\mathbb{P}$ *is a semiautomaton with an infinite state set* $\mathbb{N}_0^Q$, *the initial state* 0 *and a state transition relation* $\sqcup\!\sqcup_{\mathbb{P}} \subset \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q$ *defined by* $\sqcup\!\sqcup_{\mathbb{P}} := \{(f, \wedge(\hat{a}), g) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid (f, \hat{a}, g) \in \hat{\sqcup\!\sqcup}_{\mathbb{P}}\}$.

Adopting the notions of $\hat{S}$-automata, $A_{\mathbb{P}} \subset \sqcup\!\sqcup_{\mathbb{P}}^*$ denotes the set of all paths in $\mathbb{P}_{\sqcup\!\sqcup}$ starting with the initial state 0 and including the empty path ε. For $w \in A_{\mathbb{P}}$, $Z_{\mathbb{P}}(w)$ denotes the final state of the path w and $Z_{\mathbb{P}}(\varepsilon) := 0$. Formally the prefix closed language $A_{\mathbb{P}}$ and the function $Z_{\mathbb{P}} : A_{\mathbb{P}} \to \mathbb{N}_0^Q$ are defined inductively by

$$\varepsilon \in A_{\mathbb{P}},\ Z_{\mathbb{P}}(\varepsilon) := 0,\ w(f,a,g) \in A_{\mathbb{P}} \text{ and } Z_{\mathbb{P}}(w(f,a,g)) := g \tag{48}$$

for $w \in A_{\mathbb{P}}$, $Z_{\mathbb{P}}(w) = f$ and $(f,a,g) \in \sqcup\!\sqcup_{\mathbb{P}}$.

Let the function $\alpha_{\mathbb{P}} : A_{\mathbb{P}} \to \Sigma^*$ be inductively defined by

$$\alpha_{\mathbb{P}}(\varepsilon) := \varepsilon \text{ and } \alpha_{\mathbb{P}}(w(f,a,g)) := \alpha_{\mathbb{P}}(w)a \tag{49}$$

for $w(f,a,g) \in A_{\mathbb{P}}$ and $(f,a,g) \in \sqcup\!\sqcup_{\mathbb{P}}$. $\alpha_{\mathbb{P}}(u)$ is called the *label of a path u*.

To formally capture the relation between $\hat{\mathbb{P}}_{\sqcup\!\sqcup}$ and $\mathbb{P}_{\sqcup\!\sqcup}$, we consider the homomorphism

$$\wedge_{\mathbb{P}} : \hat{\sqcup\!\sqcup}_{\mathbb{P}}^* \to \sqcup\!\sqcup_{\mathbb{P}}^* \text{ with } \wedge_{\mathbb{P}}((f,\hat{a},g)) := (f,\wedge(\hat{a}),g) \text{ for } (f,\hat{a},g) \in \hat{\sqcup\!\sqcup}_{\mathbb{P}}. \tag{50}$$

This definition implies

$$\wedge_{\mathbb{P}} \text{ is strictly alphabetic and surjective.} \tag{51a}$$

$$\wedge_{\mathbb{P}}(y) \in A_{\mathbb{P}} \text{ iff } y \in \hat{A}_{\mathbb{P}} \text{ for } y \in \hat{\sqcup\!\sqcup}_{\mathbb{P}}^*. \tag{51b}$$

$$\hat{Z}_{\mathbb{P}}(x) = Z_{\mathbb{P}}(\wedge_{\mathbb{P}}(x)) \text{ for } x \in \hat{A}_{\mathbb{P}}. \tag{51c}$$

$$\wedge(\hat{\alpha}_{\mathbb{P}}(x)) = \alpha_{\mathbb{P}}(\wedge_{\mathbb{P}}(x)) \text{ for } x \in \hat{A}_{\mathbb{P}}. \tag{51d}$$

Now the composition of $\hat{c}_{\mathbb{P}}$ with $\wedge_{\mathbb{P}}$ attunes $\hat{c}_{\mathbb{P}}$ to $\mathbb{P}_{\sqcup\!\sqcup}$.

Definition 17.
Let the function $c_{\mathbb{P}} : \bigcap\limits_{t \in N} (\hat{\tau}_t^N)^{-1}(\text{pre}(\langle P \rangle)) \to A_{\mathbb{P}}$ *be defined by* $c_{\mathbb{P}} := \wedge_{\mathbb{P}} \circ \hat{c}_{\mathbb{P}}$.

By Corollary 3 and (51a) - (51d), Corollary 4 and Corollary 5 imply the following automata representations:

Corollary 6. $(\text{pre}(P))^{\sqcup\!\sqcup} = \alpha_{\mathbb{P}}(A_{\mathbb{P}})$ *and* $P^{\sqcup\!\sqcup} = \alpha_{\mathbb{P}}(Z_{\mathbb{P}}^{-1}(0))$.

For use in the next section the following theorem assembles the properties of the function $c_{\mathbb{P}}$, which follow from (51a) - (51d), Theorem 9 and Theorem 10:

Theorem 11. *Let* $x \in \bigcap_{t\in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))$ *then*

$$Z_{\mathbb{P}}(c_{\mathbb{P}}(x)) = \hat{n}_{\mathbb{P}}(x), \tag{52a}$$

$$\alpha_{\mathbb{P}}(c_{\mathbb{P}}(x)) = \wedge(\hat{\Theta}^N(x)), \tag{52b}$$

$$|c_{\mathbb{P}}(x)| = |x|, \tag{52c}$$

$$\mathrm{pre}(c_{\mathbb{P}}(x)) = c_{\mathbb{P}}(\mathrm{pre}(x)),\ \textit{and} \tag{52d}$$

$$c_{\mathbb{P}}[x(x^{-1}[\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))])] = c_{\mathbb{P}}(x)[(c_{\mathbb{P}}(x))^{-1}(A_{\mathbb{P}})], \tag{52e}$$

which implies

$$c_{\mathbb{P}}[\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))] = A_{\mathbb{P}}. \tag{52f}$$

5 Shuffle Projection in Terms of S-Automata

To express shuffle projection in terms of S-automata we first consider shuffle projection w.r.t. prefix closed languages. Let therefore $P,V \subset \Sigma^*$, $P \neq \emptyset$ and let $\mathbb{P}$ be an automaton for P as in Section 4. By Corollary 2 together with Theorem 6 holds $\mathrm{SP}(\mathrm{pre}(P),V)$ iff there exists an infinite countable set N such that

$$\hat{\Pi}^N_{N\setminus\{r\}}[(\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle)))\cap(\hat{\Theta}^N)^{-1}(\wedge^{-1}(V))] \subset (\hat{\Theta}^N)^{-1}(\wedge^{-1}(V)) \tag{53}$$

for each $r \in N$.
The same argument as to prove (20) shows that (53) is equivalent to

$$\hat{\Pi}^N_{N\setminus\{r\}}[(\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle)))\cap(\hat{\Theta}^N)^{-1}(\wedge^{-1}(V))] \subset$$
$$(\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle)))\cap(\hat{\Theta}^N)^{-1}(\wedge^{-1}(V)) \tag{54}$$

for each $r \in N$.
Condition (54) is a saturation property of $(\wedge \circ \hat{\Theta}^N_{|\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))})^{-1}(V)$ wrt. a binary relation on $\bigcap_{t\in N}(\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))$ defined by the homomorphisms $\hat{\Pi}^N_{N\setminus\{r\}}$ for $r \in N$. More precisely:

Let $R \subset F \times F$ be a binary relation on a set F and let $W \subset F$. The *saturation property* $\mathrm{S}(W,R)$ let be defined by

$$\mathrm{S}(W,R) \text{ iff } x \in W \text{ and } (x,y) \in R \text{ imply } y \in W. \tag{55}$$

Let $f : F \to G$, $g : G \to H$ and $V \subset H$, then (55) immediately implies

$$\mathrm{S}((g\circ f)^{-1}(V),R) \text{ iff } \mathrm{S}((g^{-1}(V),(f\otimes f)(R)), \tag{56}$$

where $f \otimes f : F \times F \to G \times G$ is defined by $(f \otimes f)((x,y)) := (f(x), f(y))$ for $(x,y) \in F \times F$.

Definition 18.

$$\text{Let } \mathcal{R}_P := \{(x,y) \in \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle)) \times \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle)) | \text{ there exists } r \in N \text{ with } y = \hat{\Pi}^N_{N \setminus \{r\}}(x)\}.$$

Now by (53) and (54)

$$\mathrm{SP}(\mathrm{pre}(P), V) \text{ iff } \mathrm{S}((\wedge \circ \hat{\Theta}^N_{|\bigcap\limits_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle))})^{-1}(V), \mathcal{R}_P). \tag{57}$$

On account of (52b) holds $\wedge \circ \hat{\Theta}^N_{|\bigcap\limits_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle))} = \alpha_{\mathbb{P}} \circ c_{\mathbb{P}}$. Therefore (56) and (57) imply

$$\mathrm{SP}(\mathrm{pre}(P), V) \text{ iff } \mathrm{S}(\alpha_{\mathbb{P}}^{-1}(V), (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathcal{R}_P)). \tag{58}$$

In Section 4 the idea to define $\mathbb{P}_{\sqcup\!\sqcup}$ was the following: Each computation in $\mathbb{P}_{\sqcup\!\sqcup}$ "correspond" to a "shuffled run" of "elementary computations". Now we will show that $(u,v) \in (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathcal{R}_P) \subset A_{\mathbb{P}} \times A_{\mathbb{P}}$ iff the "shuffled run" v' of "elementary computations" is generated from the "shuffled run" u' of "elementary computations" by "deleting" one of the "elementary computations" in u', where u "correspond" to u' and v "correspond" to v'. The formalization of this idea will result in a characterization of $(c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathcal{R}_P) \subset A_{\mathbb{P}} \times A_{\mathbb{P}}$ without explicit use of $\mathcal{R}_P$.

First we have to formalize "elementary computations": For each $r \in N$ holds $(\hat{\tau}_r^{\{r\}})^{-1}(\mathrm{pre}(\langle P \rangle)) \subset \bigcap\limits_{t \in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P \rangle))$ and $c_{\mathbb{P}}((\hat{\tau}_r^{\{r\}})^{-1}(\mathrm{pre}(\langle P \rangle))) = c_{\mathbb{P}}((\hat{\tau}_s^{\{s\}})^{-1}(\mathrm{pre}(\langle P \rangle)))$ for each $s \in N$. Therefore the following definition does not depend on $r \in N$.

Definition 19.
Let $r \in N$. The prefix closed set $E_{\mathbb{P}} := c_{\mathbb{P}}((\hat{\tau}_r^{\{r\}})^{-1}(\mathrm{pre}(\langle P \rangle))) \subset A_{\mathbb{P}}$ is called the set of elementary computations *in $\mathbb{P}_{\sqcup\!\sqcup}$.*

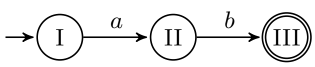


Fig. 5. Automaton $\mathbb{P}$ recognizing $P = \{ab\}$

Example 8.
Let P and $\mathbb{P}$ be defined as in Fig. 5, then $E_{\mathbb{P}} = \mathrm{pre}(\{(0, a, 1_{\mathrm{II}})(1_{\mathrm{II}}, b, 0)\})$.

$E_{\mathbb{P}}$ can also be characterized without referring to $c_{\mathbb{P}}$:

$$\begin{aligned}&E_{\mathbb{P}} = \mathrm{pre}(\{c \in Z_{\mathbb{P}}^{-1}(0) \cap \alpha_{\mathbb{P}}^{-1}(P) | Z_{\mathbb{P}}(c') = 1_{\delta(q_0, \alpha_{\mathbb{P}}(c'))} \text{ for each}\\ &c' \in \mathrm{pre}(c) \text{ with } 0 < |c'| < |c|\}), \text{ which implies}\\ &\alpha_{\mathbb{P}}(E_{\mathbb{P}}) = \mathrm{pre}(P).\end{aligned} \tag{59}$$

To formally define shuffled runs and corresponding representations, let $\check{\Sigma}$ be a disjoint copy of Σ and $\check{\imath} : \check{\Sigma}^* \to \Sigma^*$ the corresponding isomorphism. This isomorphism defines a deterministic automaton $\check{\mathbb{P}}$ isomorphic to $\mathbb{P}$ with the same state set as $\mathbb{P}$ and recognizing $\check{\imath}^{-1}(P)$. More precisely: Let $\check{\mathbb{P}} := (\check{\Sigma}, Q, \check{\delta}, q_0, F)$, where $\mathbb{P} = (\Sigma, Q, \delta, q_0, F)$ and $\check{\delta}(p, \check{a}) := \delta(p, \check{\imath}(\check{a}))$ for $\check{a} \in \check{\Sigma}$ and $p \in Q$. This definition implies

$$(f, \check{a}, g) \in ⧢_{\check{\mathbb{P}}} \text{ iff } (f, \check{\imath}(\check{a}), g) \in ⧢_{\mathbb{P}} \text{ for } f, g \in \mathbb{N}_0^Q \text{ and } \check{a} \in \check{\Sigma}. \tag{60}$$

Therefore

$$\check{\imath}_{⧢_{\check{\mathbb{P}}}}((f, \check{a}, g)) := (f, \check{\imath}(\check{a}), g) \text{ for } (f, \check{a}, g) \in ⧢_{\check{\mathbb{P}}} \tag{61}$$

defines an isomorphism $\check{\imath}_{⧢_{\check{\mathbb{P}}}} : ⧢_{\check{\mathbb{P}}}^* \to ⧢_{\mathbb{P}}^*$ with

$$\check{\imath}_{⧢_{\check{\mathbb{P}}}}(A_{\check{\mathbb{P}}}) = A_{\mathbb{P}}, \tag{62a}$$

$$\check{\imath}_{⧢_{\check{\mathbb{P}}}}(E_{\check{\mathbb{P}}}) = E_{\mathbb{P}}, \tag{62b}$$

$$Z_{\check{\mathbb{P}}} = Z_{\mathbb{P}} \circ \check{\imath}_{⧢_{\check{\mathbb{P}}} | A_{\check{\mathbb{P}}}}, \text{ and} \tag{62c}$$

$$\check{\imath} \circ \alpha_{\check{\mathbb{P}}} = \alpha_{\mathbb{P}} \circ \check{\imath}_{⧢_{\check{\mathbb{P}}} | A_{\check{\mathbb{P}}}}. \tag{62d}$$

Because of $\check{\Sigma} \cap \Sigma = \emptyset$, it also holds $⧢_{\check{\mathbb{P}}} \cap ⧢_{\mathbb{P}} = \emptyset$.

Let therefore $\pi_{⧢_{\mathbb{P}}} : (⧢_{\mathbb{P}} \dot{\cup} ⧢_{\check{\mathbb{P}}})^* \to ⧢_{\mathbb{P}}^*$ be defined by

$$\pi_{⧢_{\mathbb{P}}}(y) := y \text{ for } y \in ⧢_{\mathbb{P}} \text{ and } \pi_{⧢_{\mathbb{P}}}(y) := \varepsilon \text{ for } y \in ⧢_{\check{\mathbb{P}}}.$$

In the same way let $\pi_{⧢_{\check{\mathbb{P}}}} : (⧢_{\mathbb{P}} \dot{\cup} ⧢_{\check{\mathbb{P}}})^* \to ⧢_{\check{\mathbb{P}}}^*$ be defined by

$$\pi_{⧢_{\check{\mathbb{P}}}}(y) := \varepsilon \text{ for } y \in ⧢_{\mathbb{P}} \text{ and } \pi_{⧢_{\check{\mathbb{P}}}}(y) := y \text{ for } y \in ⧢_{\check{\mathbb{P}}}.$$

As $A_{\mathbb{P}} \subset ⧢_{\mathbb{P}}^*$ and $E_{\check{\mathbb{P}}} \subset ⧢_{\check{\mathbb{P}}}^*$ are prefix closed languages, $\pi_{⧢_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{⧢_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}}) \subset (⧢_{\mathbb{P}} \dot{\cup} ⧢_{\check{\mathbb{P}}})^*$ is also a prefix closed language. Its elements are called *shuffled runs* of a computation in $\mathbb{P}$ and an elementary computation in $\check{\mathbb{P}}$. Let now $\beta_{\mathbb{P}} : (⧢_{\mathbb{P}} \dot{\cup} ⧢_{\check{\mathbb{P}}})^* \to \Sigma^*$ be defined by

$$\begin{aligned}&\beta_{\mathbb{P}}((f, x, g)) := x \text{ for } (f, x, g) \in ⧢_{\mathbb{P}} \text{ and}\\ &\beta_{\mathbb{P}}((f, x, g)) := \check{\imath}(x) \text{ for } (f, x, g) \in ⧢_{\check{\mathbb{P}}}.\end{aligned} \tag{63}$$

A shuffled run $b \in \pi^{-1}_{\sqcup\!\sqcup_{\mathbb{P}}}(A_{\mathbb{P}}) \cap \pi^{-1}_{\sqcup\!\sqcup_{\check{\mathbb{P}}}}(E_{\check{\mathbb{P}}})$ is called a *shuffled representation* of $c \in A_{\mathbb{P}}$ by $d \in A_{\mathbb{P}}$ and $e \in E_{\check{\mathbb{P}}}$ iff

$$\alpha_{\mathbb{P}}(c) = \beta_{\mathbb{P}}(b), \tag{64a}$$

$$\pi_{\sqcup\!\sqcup_{\check{\mathbb{P}}}}(b) = e, \tag{64b}$$

$$\pi_{\sqcup\!\sqcup_{\mathbb{P}}}(b) = d, \text{ and} \tag{64c}$$

$$Z_{\mathbb{P}}(c') = Z_{\check{\mathbb{P}}}(\pi_{\sqcup\!\sqcup_{\check{\mathbb{P}}}}(b')) + Z_{\mathbb{P}}(\pi_{\sqcup\!\sqcup_{\mathbb{P}}}(b'))$$
$$\text{for each } c' \in \mathrm{pre}(c), \text{ where } b' \in \mathrm{pre}(b) \text{ with } |b'| = |c'|. \tag{64d}$$

Example 9.
Let P and $\mathbb{P}$ be defined as in Fig. 5, and

$$d = (0,a,1_{\mathrm{II}})(1_{\mathrm{II}},b,0)(0,a,1_{\mathrm{II}})(1_{\mathrm{II}},b,0) \in A_{\mathbb{P}},$$

$$e = (0,\check{a},1_{\mathrm{II}})(1_{\mathrm{II}},\check{b},0) \in E_{\check{\mathbb{P}}},$$

$$b = (0,a,1_{\mathrm{II}})(0,\check{a},1_{\mathrm{II}})(1_{\mathrm{II}},b,0)(0,a,1_{\mathrm{II}})(1_{\mathrm{II}},\check{b},0)(1_{\mathrm{II}},b,0) \in \pi^{-1}_{\sqcup\!\sqcup_{\mathbb{P}}}(A_{\mathbb{P}}) \cap \pi^{-1}_{\sqcup\!\sqcup_{\check{\mathbb{P}}}}(E_{\check{\mathbb{P}}})$$

and

$$c = (0,a,1_{\mathrm{II}})(1_{\mathrm{II}},a,2_{\mathrm{II}})(2_{\mathrm{II}},b,1_{\mathrm{II}})(1_{\mathrm{II}},a,2_{\mathrm{II}})(2_{\mathrm{II}},b,1_{\mathrm{II}})(1_{\mathrm{II}},b,0) \in A_{\mathbb{P}},$$

then b is a shuffled representation of c by d and e.

The shuffled representations define a relation $\mathcal{R}_{\mathbb{P}} \subset A_{\mathbb{P}} \times A_{\mathbb{P}}$:

Definition 20.
$\mathcal{R}_{\mathbb{P}} := \{(c,d) \in A_{\mathbb{P}} \times A_{\mathbb{P}} |$ *there exists* $e \in E_{\check{\mathbb{P}}}$ *and a shuffled representation* $b \in \pi^{-1}_{\sqcup\!\sqcup_{\mathbb{P}}}(A_{\mathbb{P}}) \cap \pi^{-1}_{\sqcup\!\sqcup_{\check{\mathbb{P}}}}(E_{\check{\mathbb{P}}})$ *of* c *by* d *and* $e\}$.

Now we will prove $\mathcal{R}_{\mathbb{P}} = (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathcal{R}_P)$. For this purpose we define an appropriate function $b_{\mathbb{P}} : N \times \bigcap_{t \in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P \rangle)) \to \pi^{-1}_{\sqcup\!\sqcup_{\mathbb{P}}}(A_{\mathbb{P}}) \cap \pi^{-1}_{\sqcup\!\sqcup_{\check{\mathbb{P}}}}(E_{\check{\mathbb{P}}})$. For it we first need a unique factorization property of the elements of $\bigcap_{t \in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P \rangle))$:

Let $w \in \bigcap_{t \in N} (\hat{\tau}^N_t)^{-1}(\mathrm{pre}(\langle P \rangle))$, $r \in N$, $x = \hat{\Pi}^N_{\{r\}}(w)$ and $y = \hat{\Pi}^N_{N \setminus \{r\}}(w)$. Then there exists exactly one $y_0 \in \hat{\Sigma}^*_{N \setminus \{r\}}$, and for each $i \in \{i \in \mathbb{N} | 1 \le i \le |x|\}$ exactly one $x_i \in \hat{\Sigma}_{\{r\}}$ as well as exactly one $y_i \in \hat{\Sigma}^*_{N \setminus \{r\}}$ such that

$$w = y = y_0 \text{ for } x = \varepsilon, \quad \text{and}$$
$$w = y_0 x_1 y_1 ... x_{|x|} y_{|x|},\ x = x_1 ... x_{|x|} \text{ as well as } y = y_0 y_1 ... y_{|x|} \text{ for } x \neq \varepsilon. \tag{65}$$

Because of $|c_{\mathbb{P}}(x)| = |x|$, $|c_{\mathbb{P}}(y)| = |y|$ and $\hat{\Pi}^N_{\{r\}}(w) = (\hat{\tau}^{\{r\}}_r)^{-1}(\hat{\tau}^N_r(w))$, which implies $c_{\mathbb{P}}(x) \in E_{\mathbb{P}}$, the following definition is sound:

Definition 21.
Let r, w, x, y, and the factorizations of w, x and y as in 65, then

$$b_{\mathbb{P}} : N \times \bigcap_{t\in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle)) \to \pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}}) \text{ is defined by}$$

$$b_{\mathbb{P}}((r,w)) := c_{\mathbb{P}}(y) \text{ for } x = \varepsilon \text{ and } b_{\mathbb{P}}((r,w)) := v_0u_1v_1...u_{|x|}v_{|x|} \text{ for } x \neq \varepsilon,$$

where $u_1...u_{|x|} = \check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(c_{\mathbb{P}}(x))$, $v_0v_1...v_{|x|} = c_{\mathbb{P}}(y)$, $|u_i| = |x_i|$ *and* $|v_k| = |y_k|$ *for* $1 \leq i \leq |x|$ *and* $0 \leq k \leq |x|$.

By this definition $c_{\mathbb{P}}(y)$ and $\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(c_{\mathbb{P}}(x))$ are shuffled in $b_{\mathbb{P}}((r,w))$ in the same manner as y and x are shuffled in w, which implies

$$|b_{\mathbb{P}}((r,w))| = |w|, \tag{66}$$

and moreover

$$\begin{aligned} &b_{\mathbb{P}}((r,w)) \in \pi_{\sqcup_{\mathbb{P}}}^{-1}(c_{\mathbb{P}}(\hat{\Pi}_{N\setminus\{r\}}^N(w))) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(c_{\mathbb{P}}(\hat{\Pi}_{\{r\}}^N(w)))),\\ &|\hat{\Pi}_{N\setminus\{r\}}^N(w')| = |\pi_{\sqcup_{\mathbb{P}}}(b')| \text{ and } |\hat{\Pi}_{\{r\}}^N(w')| = |\pi_{\sqcup_{\check{\mathbb{P}}}}(b')|\\ &\text{for each } w' \in \mathrm{pre}(w) \text{ and } b' \in \mathrm{pre}(b_{\mathbb{P}}((r,w))) \text{ with } |w'| = |b'|. \end{aligned} \tag{67}$$

It is easy to see that 67 characterizes $b_{\mathbb{P}}((r,w))$. More precisely:

$$\begin{aligned} \{b_{\mathbb{P}}((r,w))\} = \{b \in &\pi_{\sqcup_{\mathbb{P}}}^{-1}(c_{\mathbb{P}}(\hat{\Pi}_{N\setminus\{r\}}^N(w))) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(c_{\mathbb{P}}(\hat{\Pi}_{\{r\}}^N(w)))) \mid\\ &|\hat{\Pi}_{N\setminus\{r\}}^N(w')| = |\pi_{\sqcup_{\mathbb{P}}}(b')| \text{ and } |\hat{\Pi}_{\{r\}}^N(w')| = |\pi_{\sqcup_{\check{\mathbb{P}}}}(b')|\\ &\text{for each } w' \in \mathrm{pre}(w) \text{ and } b' \in \mathrm{pre}(b) \text{ with } |w'| = |b'|\}. \end{aligned} \tag{68}$$

Now (68) and Theorem 11 together with (34), (62c), (62d) and (63) imply

$$\mathrm{pre}(b_{\mathbb{P}}((r,w))) = b_{\mathbb{P}}((r,\mathrm{pre}(w))), \tag{69}$$

$$\wedge(\hat{\Theta}^N(w)) = \beta_{\mathbb{P}}(b_{\mathbb{P}}((r,w))) \text{ and} \tag{70}$$

$$\hat{n}_{\mathbb{P}}(w) = Z_{\check{\mathbb{P}}}(\pi_{\sqcup_{\check{\mathbb{P}}}}(b_{\mathbb{P}}((r,w)))) + Z_{\mathbb{P}}(\pi_{\sqcup_{\mathbb{P}}}(b_{\mathbb{P}}((r,w)))) \tag{71}$$

To complete the list of properties of $b_{\mathbb{P}}$ we will show

$$b_{\mathbb{P}}(N \times \bigcap_{t\in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))) = \pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}}). \tag{72}$$

Proof. Proof of equation (72):

Let $b \in \pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}})$. Because of (52f), Definition 19 and (62b) there exist $y \in \bigcap_{t\in N} (\hat{\tau}_t^N)^{-1}(\mathrm{pre}(\langle P\rangle))$ and $\hat{x} \in \mathrm{pre}(\langle P\rangle)$ such that $c_{\mathbb{P}}(y) = \pi_{\sqcup_{\mathbb{P}}}(b)$ and $\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(c_{\mathbb{P}}((\hat{\tau}_s^{\{s\}})^{-1}(\hat{x}))) = \pi_{\sqcup_{\check{\mathbb{P}}}}(b)$ for each $s \in N$.

Let now $r \in N \setminus \{\kappa(y)\}$, then by the same argument as in (65) and in (67) y and $(\hat{\tau}_r^{\{r\}})^{-1}(\hat{x})$ can be shuffled in the same manner as $\pi_{\sqcup_{\mathbb{P}}}(b)$ and $\pi_{\sqcup_{\check{\mathbb{P}}}}(b)$ are shuffled in b. This result in $w \in \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\text{pre}(\langle P \rangle))$ with $(\hat{\tau}_r^{\{r\}})^{-1}(\hat{x}) = \hat{\mathit{\Pi}}_{\{r\}}^N(w)$, $y = \hat{\mathit{\Pi}}_{N \setminus \{r\}}^N(w)$, $|\hat{\mathit{\Pi}}_{N \setminus \{r\}}^N(w')| = |\pi_{\sqcup_{\mathbb{P}}}(b')|$ and $|\hat{\mathit{\Pi}}_{\{r\}}^N(w')| = |\pi_{\sqcup_{\check{\mathbb{P}}}}(b')|$ for each $w' \in \text{pre}(w)$ and $b' \in \text{pre}(b)$ with $|w'| = |b'|$. Now by (68) $b_{\mathbb{P}}((r,w)) = b$, which completes the proof of equation (72).

To prove the main theorem of this section, additionally to (66) - (72) the following characterization of equality in $A_{\mathbb{P}}$ is needed, which is an immediate consequence of the definitions in (48) and (49):

$$\begin{aligned} &\text{Let } u,v \in A_{\mathbb{P}}, \text{ then } u = v \text{ iff } \alpha_{\mathbb{P}}(u) = \alpha_{\mathbb{P}}(v) \text{ and } Z_{\mathbb{P}}(u') = Z_{\mathbb{P}}(v') \\ &\text{for each } u' \in \text{pre}(u) \text{ and } v' \in \text{pre}(v) \text{ with } |u'| = |v'|. \end{aligned} \tag{73}$$

Theorem 12. $\mathcal{R}_{\mathbb{P}} = (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathcal{R}_P)$

Proof.
Let $(w,y) \in \mathcal{R}_P$, then $w \in \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\text{pre}(\langle P \rangle))$, and there exists $r \in N$ such that $y = \hat{\mathit{\Pi}}_{N \setminus \{r\}}^N(w)$. By (67) - (71) and Theorem 11 together with (34) $b_{\mathbb{P}}((r,w))$ is a shuffled representation of $c_{\mathbb{P}}(w)$ by $c_{\mathbb{P}}(y)$ and $\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(c_{\mathbb{P}}((\hat{\tau}_s^{\{s\}})^{-1}(\hat{x})))$. Therefore $(c_{\mathbb{P}}(w), c_{\mathbb{P}}(y)) \in \mathcal{R}_{\mathbb{P}}$, which proves $(c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathcal{R}_P) \subset \mathcal{R}_{\mathbb{P}}$.

To show the contrary inclusion let $(c,d) \in \mathcal{R}_{\mathbb{P}}$. Then there exists $e \in E_{\check{\mathbb{P}}}$ and a shuffled representation $b \in \pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}})$ of c by d and e. By (72) there exists $w \in \bigcap_{t \in N} (\hat{\tau}_t^N)^{-1}(\text{pre}(\langle P \rangle))$, and $r \in N$ such that $b = b_{\mathbb{P}}((r,w))$. Now (68) - (71) and Theorem 11 together with (73) imply $(c,d) = (c_{\mathbb{P}}(w), c_{\mathbb{P}}(\hat{\mathit{\Pi}}_{N \setminus \{r\}}^N(w))) = (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(w, \hat{\mathit{\Pi}}_{N \setminus \{r\}}^N(w)) \in (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathcal{R}_P)$. Therefore $\mathcal{R}_{\mathbb{P}} \subset (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathcal{R}_P)$, which completes the proof of Theorem 12.

Now we consider shuffle projections w.r.t. arbitrary languages. Therefore in Definition 18 $\text{pre}(\langle P \rangle)$ has to be replaced by $\langle P \rangle \cup \{\varepsilon\}$. So on account of (46) we define:

Definition 22.
Let $\mathring{\mathcal{R}}_P := \{(x,y) \in \hat{n}_{\mathbb{P}}^{-1}(0) \times \hat{n}_{\mathbb{P}}^{-1}(0) | \textit{there exists } r \in N \textit{ with } y = \hat{\mathit{\Pi}}_{N \setminus \{r\}}^N(x)\}$.

Because of $\hat{\mathit{\Pi}}_{N \setminus \{r\}}^N(\hat{n}_{\mathbb{P}}^{-1}(0)) \subset \hat{n}_{\mathbb{P}}^{-1}(0)$ it holds

$$\mathring{\mathcal{R}}_P = \mathcal{R}_P \cap (\hat{n}_{\mathbb{P}}^{-1}(0) \times \hat{n}_{\mathbb{P}}^{-1}(0)). \tag{74}$$

Now by the same argument as in (57)

$$\text{SP}(P \cup \{\varepsilon\}, V) \text{ iff } \text{S}((\wedge \circ \hat{\Theta}_{|\hat{n}_{\mathbb{P}}^{-1}(0)}^N)^{-1}(V), \mathring{\mathcal{R}}_P). \tag{75}$$

On account of (52b) holds $\wedge \circ \hat{\Theta}^N_{|\hat{n}_{\mathbb{P}}^{-1}(0)} = \alpha_{\mathbb{P}} \circ c_{\mathbb{P}|\hat{n}_{\mathbb{P}}^{-1}(0)}$. Therefore (56) and (75) imply

$$\mathrm{SP}(P \cup \{\varepsilon\}, V) \text{ iff } \mathrm{S}(\alpha_{\mathbb{P}}^{-1}(V), (c_{\mathbb{P}|\hat{n}_{\mathbb{P}}^{-1}(0)} \otimes c_{\mathbb{P}|\hat{n}_{\mathbb{P}}^{-1}(0)})(\mathring{\mathcal{R}}_P)),$$

and because of $(c_{\mathbb{P}|\hat{n}_{\mathbb{P}}^{-1}(0)} \otimes c_{\mathbb{P}|\hat{n}_{\mathbb{P}}^{-1}(0)})(\mathring{\mathcal{R}}_P) = (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathring{\mathcal{R}}_P)$

$$\mathrm{SP}(P \cup \{\varepsilon\}, V) \text{ iff } \mathrm{S}(\alpha_{\mathbb{P}}^{-1}(V), (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathring{\mathcal{R}}_P)). \tag{76}$$

Theorem 12 allows to characterize the relation $(c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathring{\mathcal{R}}_P) \subset A_{\mathbb{P}} \times A_{\mathbb{P}}$ without explicit use of $\mathring{\mathcal{R}}_P$:

Corollary 7. $(c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathring{\mathcal{R}}_P) = \mathcal{R}_{\mathbb{P}} \cap (Z_{\mathbb{P}}^{-1}(0) \times Z_{\mathbb{P}}^{-1}(0)) =: \mathring{\mathcal{R}}_{\mathbb{P}}$.

Proof.
(52a) (74) and Theorem 12 imply

$$\begin{aligned}
(c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathring{\mathcal{R}}_P) = & (c_{\mathbb{P}} \otimes c_{\mathbb{P}})[\mathcal{R}_P \cap (\hat{n}_{\mathbb{P}}^{-1}(0) \times \hat{n}_{\mathbb{P}}^{-1}(0))] = \\
& (c_{\mathbb{P}} \otimes c_{\mathbb{P}})[\mathcal{R}_P \cap (c_{\mathbb{P}}^{-1}(Z_{\mathbb{P}}^{-1}(0)) \times c_{\mathbb{P}}^{-1}(Z_{\mathbb{P}}^{-1}(0)))] = \\
& (c_{\mathbb{P}} \otimes c_{\mathbb{P}})[\mathcal{R}_P \cap (c_{\mathbb{P}}^{-1} \otimes c_{\mathbb{P}}^{-1})(Z_{\mathbb{P}}^{-1}(0) \times Z_{\mathbb{P}}^{-1}(0))] = \\
& (c_{\mathbb{P}} \otimes c_{\mathbb{P}})(\mathcal{R}_P) \cap (Z_{\mathbb{P}}^{-1}(0) \times Z_{\mathbb{P}}^{-1}(0)) = \\
& \mathcal{R}_{\mathbb{P}} \cap (Z_{\mathbb{P}}^{-1}(0) \times Z_{\mathbb{P}}^{-1}(0)),
\end{aligned}$$

which completes the proof of Corollary 7.

Considering the powerset 2^F, a binary relation $R \subset F \times F$ defines a function $R' : 2^F \to 2^F$ by

$$R'(U) := \{y \in F | \text{there exists } x \in U \text{ with } (x,y) \in R\} \text{ for each } U \in 2^F. \tag{77}$$

It is an immediate consequence that

$$R'(U) = \bigcup_{x \in U} R'(\{x\}) \quad \text{for each } U \in 2^F. \tag{78}$$

Now,

$$\mathrm{S}(W, R) \text{ iff } R'(W) \subset W, \text{ for each } W \in 2^F. \tag{79}$$

Applying (79) to (58) and Theorem 12 result in

Corollary 8.

$$\mathrm{SP}(\mathrm{pre}(P), V) \textit{ iff } \mathcal{R}'_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(V)) \subset \alpha_{\mathbb{P}}^{-1}(V).$$

Corollary 7 implies

$$\mathring{\mathcal{R}}'_{\mathbb{P}}(U) = \mathcal{R}'_{\mathbb{P}}(U \cap Z_{\mathbb{P}}^{-1}(0)) \cap Z_{\mathbb{P}}^{-1}(0) \text{ for each } U \subset A_{\mathbb{P}}. \tag{80}$$

On account of (64d) holds

$$\mathcal{R}'_{\mathbb{P}}(Z_{\mathbb{P}}^{-1}(0)) \subset Z_{\mathbb{P}}^{-1}(0), \tag{81}$$

and therefore by (80)

$$\mathring{\mathcal{R}}'_{\mathbb{P}}(U) = \mathcal{R}'_{\mathbb{P}}(U \cap Z_{\mathbb{P}}^{-1}(0)) \text{ for each } U \subset A_{\mathbb{P}}. \tag{82}$$

Now from (76), (79), (81), and (82) it follows

Corollary 9.

$$\begin{aligned}\mathrm{SP}(P \cup \{\varepsilon\}, V) \;&\textit{iff}\; \mathcal{R}'_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0)) \subset \alpha_{\mathbb{P}}^{-1}(V)\\ &\textit{iff}\; \mathcal{R}'_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0)) \subset \alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0).\end{aligned}$$

6 Construction Principles

Under certain conditions for a fixed language P Corollary 8 allows to construct a variety of languages V such that $\mathrm{SP}(\mathrm{pre}(P), V)$. The key to such constructions is the following implication of (64d):

$$Z_{\mathbb{P}}(\mathrm{pre}(\mathcal{R}'_{\mathbb{P}}(\{c\}))) \subset \bigcup_{x \in \mathrm{pre}(c)} \{f \in \mathbb{N}_0^Q | f \leq Z_{\mathbb{P}}(x)\} \text{ for each } c \in A_{\mathbb{P}}, \tag{83}$$

where Q is the state set of $\mathbb{P}$.

Definition 23 (initial segment).
$\emptyset \neq I \subset \mathbb{N}_0^Q$ is called initial segment *iff $r \leq s \in I$ implies $r \in I$. For each initial segment I, let $A_{(I,\mathbb{P})} := \{c \in A_{\mathbb{P}} | Z_{\mathbb{P}}(\mathrm{pre}(c)) \subset I\}$.*

It holds $\emptyset \neq A_{(I,\mathbb{P})} = \mathrm{pre}(A_{(I,\mathbb{P})})$.

Definition 24.
An initial segment I is called compatible with $\mathbb{P}$ *iff $A_{(I,\mathbb{P})}$ is saturated by the partition of $A_{\mathbb{P}}$ induced by $\alpha_{\mathbb{P}}$. I.e. $c, c' \in A_{(I,\mathbb{P})}$ and $\alpha_{\mathbb{P}}(c') = \alpha_{\mathbb{P}}(c)$ implies $c' \in A_{(I,\mathbb{P})}$. For an initial segment I compatible with $\mathbb{P}$, let $L_{(I,\mathbb{P})} := \alpha_{\mathbb{P}}(A_{(I,\mathbb{P})})$.*

By this definition $\emptyset \neq L_{(I,\mathbb{P})} \subset (\mathrm{pre}(P))^{\sqcup\!\sqcup}$ and $L_{(I,\mathbb{P})} = \mathrm{pre}(L_{(I,\mathbb{P})})$.

Theorem 13. *Let $\emptyset \neq P \subset \Sigma^*$ and I an initial segment compatible with $\mathbb{P}$, then* $\mathrm{SP}(\mathrm{pre}(P), L_{(I,\mathbb{P})})$.

Proof. On account of Corollary 8 and (78) it is sufficient to show

$$\mathcal{R}'_{\mathbb{P}}(\{c\}) \subset \alpha_{\mathbb{P}}^{-1}(L_{(I,\mathbb{P})}) \text{ for each } c \in \alpha_{\mathbb{P}}^{-1}(L_{(I,\mathbb{P})}). \tag{84}$$

Since the initial segment I is compatible with $\mathbb{P}$ it holds

$$\alpha_{\mathbb{P}}^{-1}(L_{(I,\mathbb{P})}) = \{x \in A_{\mathbb{P}} | Z_{\mathbb{P}}(\mathrm{pre}(x)) \subset I\}. \tag{85}$$

Now (83) and (85) imply (84), which completes the proof of Theorem 13.

An immediate consequence of Definition 24 is

Lemma 8. *An initial segment I is compatible with $\mathbb{P}$ iff for each $c, \check{c} \in A_{(I,\mathbb{P})}$ with $\alpha(c) = \alpha(\check{c})$, and for each $(Z_{\mathbb{P}}(c), a, f) \in \sqcup_{\mathbb{P}}$ and $(Z_{\mathbb{P}}(\check{c}), a, \check{f}) \in \sqcup_{\mathbb{P}}$ holds $f \in I$ iff $\check{f} \in I$.*

The condition of Lemma 8 can be checked by a partial powerset construction on $\mathbb{P}_{\sqcup}$. For this purpose let the partial function $\mathcal{D}_{(I,\mathbb{P})} : 2^I \times \Sigma \to 2^I$ be defined by

$$\begin{aligned}
&\mathcal{D}_{(I,\mathbb{P})}(M,a) := \{f \in \mathbb{N}_0^Q |\text{ there exist } g \in M \text{ and } (g,a,f) \in \sqcup_{\mathbb{P}}\} \\
&\text{for each } (M,a) \in 2^I \times \Sigma \text{ with} \\
&\emptyset \neq \{f \in \mathbb{N}_0^Q |\text{ there exist } g \in M \text{ and } (g,a,f) \in \sqcup_{\mathbb{P}}\} \subset I.
\end{aligned} \tag{86}$$

The partial function $\mathcal{D}_{(I,\mathbb{P})}$ defines a deterministic semiautomaton

$$\mathcal{P}_{(I,\mathbb{P})} := (\Sigma, 2^I, \mathcal{D}_{(I,\mathbb{P})}, \{0\}). \tag{87}$$

Now Lemma 8 implies

Theorem 14. *An initial segment $I \subset \mathbb{N}_0^Q$ is compatible with $\mathbb{P}$, iff for each $a \in \Sigma$ and $M \in 2^I$ reachable in $\mathcal{P}_{(I,\mathbb{P})}$ either $\mathcal{D}_{(I,\mathbb{P})}(M,a)$ is defined, or $\{f \in \mathbb{N}_0^Q |$ there exist $g \in M$ and $(g,a,f) \in \sqcup_{\mathbb{P}}\} \subset \mathbb{N}_0^Q \setminus I$.*
In that case $\mathcal{P}_{(I,\mathbb{P})}$ recognizes $L_{(I,\mathbb{P})}$.

Example 10.
Let $\tilde{P} = \{abc\}$, $\tilde{\mathbb{P}}$ as defined in Fig. 6, and $\tilde{I} = \{0, 1_{II}, 1_{III}, 1_{II} + 1_{III}\}$. The partial

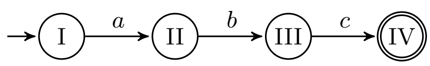


Fig. 6. Automaton $\tilde{\mathbb{P}}$ recognizing $\tilde{P}$

powerset construction result in the semiautomaton $\mathcal{P}_{(\tilde{I},\tilde{\mathbb{P}})}$ of Fig. 7, which fulfills the conditions of Theorem 14. Therefore $\tilde{I}$ is compatible with $\tilde{\mathbb{P}}$, which implies $\mathrm{SP}(\mathrm{pre}(\tilde{P}), L_{(\tilde{I},\tilde{\mathbb{P}})})$.

It is an immediate consequence of Definition 16 that

$$Z_{\mathbb{P}}(A_{\mathbb{P}}) \subset T(Q) := \{f \in \mathbb{N}_0^Q | \{q \in Q | f(q) \neq 0\} \text{ is a finite set.}\} \tag{88}$$

for each deterministic automaton $\mathbb{P}$ with state set Q (not necessarily finite).

There are special initial sections $I \subset T(Q)$ and automata $\mathbb{P}$ with state set Q, such that compatibility of I with $\mathbb{P}$ can be verified easily:

$$\text{For } f \in T(Q) \text{ let } \|f\| := \sum_{q \in Q} f(q) \in \mathbb{N}_0. \tag{89}$$

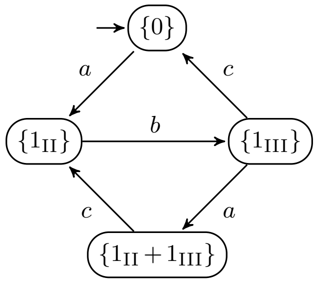


Fig. 7. Semiautomaton $\mathcal{P}_{(\tilde{I},\tilde{\mathbb{P}})}$ recognizing $L_{(\tilde{I},\tilde{\mathbb{P}})}$

$$\text{For } n \in \mathbb{N}_0 \text{ let } K(n,Q) := \{f \in T(Q) | \|f\| \leq n\}, \tag{90}$$

which is an initial segment.

Theorem 15.
Let Φ, Γ, and Ω be pairwise disjoint sets, $\emptyset \neq P \subset \Gamma \cup \Phi\Gamma^*\Omega$ and $\mathbb{P}$ be a deterministic automaton with state set Q recognizing P. Then $K(n,Q)$ is compatible with $\mathbb{P}$ for each $n \in \mathbb{N}_0$, and therefore $\mathrm{SP}(\mathrm{pre}(P), L_{(K(n,Q),\mathbb{P})})$.

Proof.
From Definition 16 it follows for each $(f,a,g) \in \sqcup_{\mathbb{P}}$

$$\begin{aligned} &a \in \Phi \text{ implies } \|g\| = \|f\| + 1, \\ &a \in \Gamma \text{ implies } \|g\| = \|f\|, \text{ and} \\ &a \in \Omega \text{ implies } \|g\| = \|f\| - 1. \end{aligned} \tag{91}$$

Therefore

$$f, f' \in M \text{ implies } \|f\| = \|f'\| \text{ for each state } M \text{ reachable in } \mathcal{P}_{(K(n,Q),\mathbb{P})}. \tag{92}$$

Now (91) and (92) together with Theorem 13 completes the proof.

Example 11.
Let $\bar{\mathbb{P}}$ and $\bar{P}$ as defined in Figure 8. Then by Theorem 15 $K(n,\bar{Q})$ is compatible with $\bar{\mathbb{P}}$ for each $n \in \mathbb{N}_0$, where $\bar{Q}$ is the state set of $\bar{\mathbb{P}}$, and it holds $\mathrm{SP}(\mathrm{pre}(\bar{P}), L_{(K(n,\bar{Q}),\bar{\mathbb{P}})})$ for each $n \in \mathbb{N}_0$.

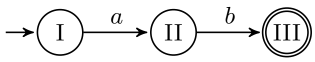


Fig. 8. Automaton $\bar{\mathbb{P}}$ recognizing $\bar{P} := \{ab\}$

Figure 9 shows the semiautomaton $\mathcal{P}_{(K(n,\bar{Q}),\bar{\mathbb{P}})}$.

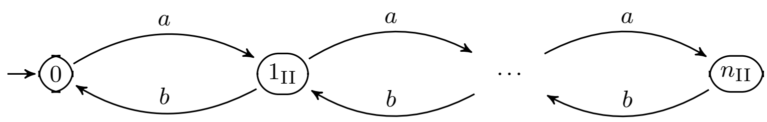


Fig. 9. Semiautomaton recognizing $L_{(K(n,\bar{Q}),\bar{\mathbb{P}})}$ for each $n \in \mathbb{N}_0$

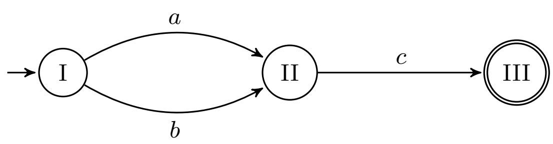


Fig. 10. Automaton $\mathring{\mathbb{P}}$ recognizing $\mathring{P}$

The following example is a bridge to the next section.

Example 12.
Let $\mathring{P}$ and $\mathring{\mathbb{P}}$ as defined in Fig. 10. It holds $Z_{\mathring{\mathbb{P}}}(A_{\mathring{\mathbb{P}}}) = \{0\} \cup \{n_{\mathrm{II}} | n \in \mathbb{N}\}$. Therefore, $ab \in L_{(\mathring{I},\mathring{\mathbb{P}})}$ implies $ba \in L_{(\mathring{I},\mathring{\mathbb{P}})}$ for each initial segment $\mathring{I}$ compatible with $\mathring{\mathbb{P}}$.

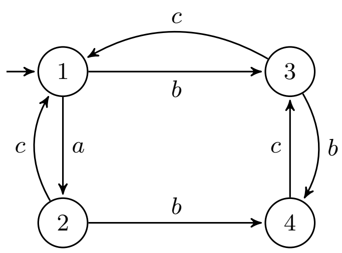


Fig. 11. Semiautomaton $\mathring{\mathbb{V}}$ recognizing $\mathring{V}$

Let the prefix closed language $\mathring{V}$ be defined by the semiautomaton in Fig. 11. Because of $ab \in \mathring{V}$ but $ba \notin \mathring{V}$, $\mathring{V}$ cannot be represented by $\mathring{V} = L_{(\mathring{I}_{\mathring{V}},\mathring{\mathbb{P}})}$ with an initial segment $\mathring{I}_{\mathring{V}}$ compatible with $\mathring{\mathbb{P}}$. So $\mathrm{SP}(\mathrm{pre}(\mathring{P}), \mathring{V})$ cannot be shown by theorem 13. But in the next section a method will be developed to prove $\mathrm{SP}(\mathrm{pre}(\mathring{P}), \mathring{V})$.

7 Representation Theorem

In this section a representation of $\mathcal{R}'_{\mathbb{P}}$ will be developed, which shows certain restrictions of $\mathcal{R}'_{\mathbb{P}}$ to be rational transductions [1]. More precisely: Depending on a subset $\Delta \subset \sqcup_{\mathbb{P}}$, an alphabet $\Delta^{()}$ and a prefix closed language $W_\Delta \subset \Delta^{()*}$

will be constructed, which represents the function $\mathcal{R}'_{\mathbb{P}|2^{A_{\mathbb{P}}\cap\Delta^*}}$ in the following manner:

There exist two alphabetic homomorphisms $\mu_\Delta : \Delta^{()*} \to \Delta^*$ and $\nu_\Delta : \Delta^{()*} \to \sqcup_{\mathbb{P}}^*$ such that for each $c \in A_{\mathbb{P}} \cap \Delta^*$ it holds,

$$d \in \mathcal{R}'_{\mathbb{P}}(\{c\}) \text{ iff there exists } x \in W_\Delta \text{ with } c = \mu_\Delta(x) \text{ and } d = \nu_\Delta(x),$$

which is equivalent to

$$\mathcal{R}'_{\mathbb{P}}(B) = \nu_\Delta(\mu_\Delta^{-1}(B) \cap W_\Delta) \text{ for each } B \subset A_{\mathbb{P}} \cap \Delta^*. \tag{93}$$

Additionally, it will be shown that

$$W_\Delta \text{ is regular if } \Delta \text{ is finite.} \tag{94}$$

In that case $\mathcal{R}'_{\mathbb{P}|2^{A_{\mathbb{P}}\cap\Delta^*}}$ is a rational transduction [1].

On account of (88) it can be assumed

$$\Delta \subset \sqcup_{\mathbb{P}} \cap (T(Q) \times \Sigma \times T(Q)). \tag{95}$$

The construction of W_Δ is based on the following idea: Each $x \in W_\Delta$ uniquely describes a shuffled representation b of $c \in A_{\mathbb{P}} \cap \Delta^*$ by $d \in A_{\mathbb{P}}$ and $e \in E_{\check{\mathbb{P}}}$ as defined in (64). This description is structured into three tracks, respectively one for c, d, and e. Additionally the second and third track describe the position of d and e in b such that both tracks together represent b. These three tracks will be formalized by three components of the elements of $\Delta^{()}$.

By an appropriate definition of $\Delta^{()}$, W_Δ can be defined as a *local* prefix closed language [1]. So W_Δ will be defined by $\Delta^{()}$, the set of initial letters of its words and the set of forbidden adjacencies of letters in its words. Generally, local languages with a finite alphabet are regular languages [1]. Starting basis for this are the definitions of $A_{\mathbb{P}} \cap \Delta^*$ and $E_{\check{\mathbb{P}}}$ as local prefix closed languages: (48) imply

$$\begin{aligned} &A_{\mathbb{P}} \cap \Delta^* = \\ &(\{\varepsilon\} \cup \{(f,a,g) \in \Delta | f = 0\}\Delta^*) \setminus \Delta^*\{(f,a,g)(f',a',g') \in \Delta\Delta | g \neq f'\}\Delta^*. \end{aligned} \tag{96}$$

With $\sqcup_{\check{\mathbb{P}}}^E := \{(f,a,g) \in \sqcup_{\check{\mathbb{P}}} | f,g \in \{0\} \cup \{1_q \in \mathbb{N}_0^Q | q \in Q\}\}$ (59) imply

$$\begin{aligned} E_{\check{\mathbb{P}}} = &(\{\varepsilon\} \cup \{(f,a,g) \in \sqcup_{\check{\mathbb{P}}}^E | f = 0\} \sqcup_{\check{\mathbb{P}}}^{E*}) \setminus \\ &\sqcup_{\check{\mathbb{P}}}^{E*} \{(f,a,g)(f',a',g') \in \sqcup_{\check{\mathbb{P}}}^E \sqcup_{\check{\mathbb{P}}}^E | g \neq f' \text{ or } g = 0\} \sqcup_{\check{\mathbb{P}}}^{E*}. \end{aligned} \tag{97}$$

To achieve (93) and (94), W_Δ has to be defined in such a way, that for each $x \in W_\Delta$ the corresponding c and d can be extracted from x by alphabetic homomorphisms, and that finiteness of Δ implies regularity of W_Δ. For formalization let

$$\Delta^{()} \subset \Delta^{()'} := \Delta^{(1)} \times \Delta^{(2)} \times \Delta^{(3)}, \tag{98}$$

where $\Delta^{(1)}$ is the alphabet for representing $c \in A_{\mathbb{P}} \cap \Delta^*$, and $\Delta^{(2)} \times \Delta^{(3)}$ is the alphabet for representing $b \in \pi_{\sqcup\!\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{\sqcup\!\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}})$, the shuffled representation of c. This is possible, since $|c| = |b|$ because of (64a).

In that representation $\Delta^{(2)}$ is the alphabet for representing $d = \pi_{\sqcup\!\sqcup_{\mathbb{P}}}(b) \in A_{\mathbb{P}}$ as well as for describing the positioning of d inside b. $\Delta^{(3)}$ is the alphabet for representing $e = \pi_{\sqcup\!\sqcup_{\check{\mathbb{P}}}}(b) \in E_{\check{\mathbb{P}}}$ as well as for describing the positioning of e inside b. Additionally it should be noticed that each $b \in (\sqcup\!\sqcup_{\mathbb{P}} \dot\cup \sqcup\!\sqcup_{\check{\mathbb{P}}})^*$ is uniquely determined by $\pi_{\sqcup\!\sqcup_{\mathbb{P}}}(b)$, $\pi_{\sqcup\!\sqcup_{\check{\mathbb{P}}}}(b)$ and by the information, which positions of b contain elements of $\sqcup\!\sqcup_{\mathbb{P}}$ and which positions contain elements of $\sqcup\!\sqcup_{\check{\mathbb{P}}}$.

As $\Delta^{(1)}$ is the alphabet for representing $c \in A_{\mathbb{P}} \cap \Delta^*$, let

$$\Delta^{(1)} := \Delta, \tag{99}$$

and let $\varphi_{\Delta}^{(1)} : \Delta^{()'*} \to \Delta^{(1)*}$ be the homomorphism defined by

$$\varphi_{\Delta}^{(1)}((x_1,x_2,x_3)) := x_1 \text{ for } (x_1,x_2,x_3) \in \Delta^{()'}. \tag{100}$$

Then $\varphi_{\Delta}^{(1)}(x) \in A_{\mathbb{P}} \cap \Delta^*$ should hold for each $x \in W_{\Delta}$.

Let now the mappings $\varphi_{\Delta}^{(1.1)}$, $\varphi_{\Delta}^{(1.2)}$ and $\varphi_{\Delta}^{(1.3)}$ be defined by

$$\varphi_{\Delta}^{(1.1)} : \Delta^{(1)} \to \mathbb{N}_0^Q \text{ with } \varphi_{\Delta}^{(1.1)}((f,a,g)) := f, \tag{101}$$

$$\varphi_{\Delta}^{(1.2)} : \Delta^{(1)} \to \Sigma \text{ with } \varphi_{\Delta}^{(1.2)}((f,a,g)) := a, \tag{102}$$

and

$$\varphi_{\Delta}^{(1.3)} : \Delta^{(1)} \to \mathbb{N}_0^Q \text{ with } \varphi_{\Delta}^{(1.3)}((f,a,g)) := g \tag{103}$$

for each $(f,a,g) \in \Delta^{(1)}$.

Then (96) becomes

$$A_{\mathbb{P}} \cap \Delta^* = (\{\varepsilon\} \cup (\varphi_{\Delta}^{(1.1)})^{-1}(0)\Delta^{(1)*}) \setminus \Delta^{(1)*}F^{(1)}\Delta^{(1)*}$$

with

$$F^{(1)} := \{xy \in \Delta^{(1)}\Delta^{(1)} | \varphi_{\Delta}^{(1.3)}(x) \neq \varphi_{\Delta}^{(1.1)}(y)\}. \tag{104}$$

Therefore $\varphi_{\Delta}^{(1)}(W_{\Delta}) \subset A_{\mathbb{P}} \cap \Delta^*$ if

$$\varphi_{\Delta}^{(1)}(W_{\Delta}) \subset (\{\varepsilon\} \cup (\varphi_{\Delta}^{(1.1)})^{-1}(0)\Delta^{(1)*}) \setminus \Delta^{(1)*}F^{(1)}\Delta^{(1)*}. \tag{105}$$

With two further conditions similar to (105) and additional restrictions of the alphabet $\Delta^{()'}$ the language W_{Δ} will be defined. But first the sets $\Delta^{(2)}$ and $\Delta^{(3)}$ have to be defined.

Since the elements of W_Δ particularly have to represent condition (64d) let

$$S_\Delta^{(1)} := Z_{\mathbb{P}}(A_{\mathbb{P}} \cap \Delta^*), \tag{106}$$

$$S_\Delta^{(3)} := \{f \in Z_{\check{\mathbb{P}}}(E_{\check{\mathbb{P}}}) | \text{ there exists } g \in S_\Delta^{(1)} \text{ with } f \leq g\}, \tag{107}$$

and

$$S_\Delta^{(2)} := \{f \in Z_{\mathbb{P}}(A_{\mathbb{P}}) | \text{ there exists } g \in S_\Delta^{(1)} \text{ and } h \in S_\Delta^{(3)} \text{ with } g = f + h\}. \tag{108}$$

Now, on account of (95)

$$\begin{aligned}&\text{Finiteness of } \Delta \text{ implies finiteness of } S_\Delta^{(1)}, S_\Delta^{(2)} \text{ and of } S_\Delta^{(3)},\\ &\text{which can be effectively determined.}\end{aligned} \tag{109}$$

With $\Sigma_\Delta := \varphi_\Delta^{(1.2)}(\Delta^{(1)}) = \varphi_\Delta^{(1.2)}(\Delta) \subset \Sigma$

$$\text{finiteness of } \Delta \text{ implies finiteness of } \Sigma_\Delta. \tag{110}$$

By the definition

$$\Delta^{(2)'} := \sqcup_{\mathbb{P}} \cap (S_\Delta^{(2)} \times \Sigma_\Delta \times S_\Delta^{(2)}) \tag{111}$$

holds $d \in A_{\mathbb{P}} \cap \Delta^{(2)'*}$ for $d = \pi_{\sqcup_{\mathbb{P}}}(b)$ because of (64d).

Now $S_\Delta^{(2)}$ is used to describe the positioning of d inside b. Let therefore

$$\begin{aligned}&\Delta^{(2)} := \Delta^{(2)'} \cup S_\Delta^{(2)}, \text{ which is finite if } \Delta \text{ is finite,}\\ &\text{and can be effectively determined.}\end{aligned} \tag{112}$$

Let the homomorphisms $\varphi_\Delta^{(2)} : \Delta^{()'*} \to \Delta^{(2)*}$ and $\gamma_\Delta^{(2)} : \Delta^{(2)*} \to \Delta^{(2)'*}$ be defined by

$$\begin{aligned}&\varphi_\Delta^{(2)}((x_1, x_2, x_3)) := x_2 \text{ for } (x_1, x_2, x_3) \in \Delta^{()'},\\ &\gamma_\Delta^{(2)}(y) := y \text{ for } y \in \Delta^{(2)'} \text{ and}\\ &\gamma_\Delta^{(2)}(y) := \varepsilon \text{ for } y \in S_\Delta^{(2)}.\end{aligned} \tag{113}$$

Now, on account of (64c) $\gamma_\Delta^{(2)}(\varphi_\Delta^{(2)}(x)) \in A_{\mathbb{P}} \cap \Delta^{(2)'*}$ should hold for each $x \in W_\Delta$.

With the mappings $\varphi_\Delta^{(2.1)} : \Delta^{(2)} \to S_\Delta^{(2)}$ and $\varphi_\Delta^{(2.3)} : \Delta^{(2)} \to S_\Delta^{(2)}$ defined by

$$\begin{aligned}&\varphi_\Delta^{(2.1)}((f, a, g)) := f \text{ and } \varphi_\Delta^{(2.3)}((f, a, g)) := g \text{ for } (f, a, g) \in \Delta^{(2)'} \text{ and}\\ &\varphi_\Delta^{(2.1)}(f) := \varphi_\Delta^{(2.3)}(f) := f \text{ for } f \in S_\Delta^{(2)}\end{aligned} \tag{114}$$

it holds $\gamma_\Delta^{(2)}(\varphi_\Delta^{(2)}(W_\Delta)) \subset A_{\mathbb{P}} \cap \Delta^{(2)'*}$ if

$$\varphi_\Delta^{(2)}(W_\Delta) \subset (\{\varepsilon\} \cup (\varphi_\Delta^{(2.1)})^{-1}(0)\Delta^{(2)*}) \setminus \Delta^{(2)*}F^{(2)}\Delta^{(2)*} \text{ where}$$
$$F^{(2)} := \{xy \in \Delta^{(2)}\Delta^{(2)} | \varphi_\Delta^{(2.3)}(x) \neq \varphi_\Delta^{(2.1)}(y)\}. \tag{115}$$

Let the mapping $Z_\Delta^{(2)} : \Delta^{(2)*} \to S_\Delta^{(2)}$ be defined by

$$Z_\Delta^{(2)}(\varepsilon) := 0, \text{ and } Z_\Delta^{(2)}(uv) := \varphi_\Delta^{(2.3)}(v) \text{ for } u \in \Delta^{(2)*} \text{ and } v \in \Delta^{(2)}. \tag{116}$$

Then (115) implies

$$Z_\Delta^{(2)}(\varphi_\Delta^{(2)}(x)) = Z_{\mathbb{P}}(\gamma_\Delta^{(2)}(\varphi_\Delta^{(2)}(x))) \text{ for each } x \in W_\Delta. \tag{117}$$

Now, the definitions concerning $\Delta^{(3)}$ are similar to those concerning $\Delta^{(2)}$. But additionally it must be pointed out that

$$E_{\check{\mathbb{P}}} \subset \sqcup_{\check{\mathbb{P}}}^{E*} \setminus (\sqcup_{\check{\mathbb{P}}}^{E*}\{(f,a,g)(f',a',g') \in \sqcup_{\check{\mathbb{P}}}^{E} \sqcup_{\check{\mathbb{P}}}^{E} | g = 0\}\sqcup_{\check{\mathbb{P}}}^{E*}).$$

Therefore we use an additional letter $\check{0} \notin \sqcup_{\check{\mathbb{P}}}^{E} \cup S_\Delta^{(3)}$ to define the content of the third track by a prefix closed local language such that

$$\varphi_\Delta^{(3)}(W_\Delta) \subset \mathrm{pre}((S_\Delta^{(3)} \cup \{(f,a,g) \in \sqcup_{\check{\mathbb{P}}}^{E} | g \neq 0\})^*\{(f,a,g) \in \sqcup_{\check{\mathbb{P}}}^{E} | g = 0\}\{\check{0}\}^*).$$

So let

$$\Delta^{(3)'} := \sqcup_{\check{\mathbb{P}}}^{E} \cap (S_\Delta^{(3)} \times \check{\iota}^{-1}(\Sigma_\Delta) \times S_\Delta^{(3)}) \text{ and } \Delta^{(3)} := \Delta^{(3)'} \cup S_\Delta^{(3)} \cup \{\check{0}\},$$
$$\text{which are finite and can be effectively determined, if } \Delta \text{ is finite.} \tag{118}$$

By this definition of $\Delta^{(3)'}$ holds $e \in E_{\check{\mathbb{P}}} \cap \Delta^{(3)'*}$ for $e = \pi_{\sqcup_{\check{\mathbb{P}}}}(b)$ because of (64d). $S_\Delta^{(3)} \cup \{\check{0}\}$ is used to describe the positioning of e inside b.

Let the homomorphisms $\varphi_\Delta^{(3)} : \Delta^{()'*} \to \Delta^{(3)*}$ and $\gamma_\Delta^{(3)} : \Delta^{()*} \to \Delta^{(3)'*}$ be defined by

$$\varphi_\Delta^{(3)}((x_1,x_2,x_3)) := x_3 \text{ for } (x_1,x_2,x_3) \in \Delta^{()'},$$
$$\gamma_\Delta^{(3)}(y) := y \text{ for } y \in \Delta^{(3)'} \text{ and}$$
$$\gamma_\Delta^{(3)}(y) := \varepsilon \text{ for } y \in S_\Delta^{(3)} \cup \{\check{0}\}. \tag{119}$$

Now, on account of (64b) $\gamma_\Delta^{(3)}(\varphi_\Delta^{(3)}(x)) \in E_{\check{\mathbb{P}}} \cap \Delta^{(3)'*}$ should hold for each $x \in W_\Delta$.

With the mappings $\varphi_\Delta^{(3.1)} : \Delta^{(3)} \to S_\Delta^{(3)}$ and $\varphi_\Delta^{(3.3)} : \Delta^{(3)} \to S_\Delta^{(3)}$ defined by

$$\varphi_\Delta^{(3.1)}((f,a,g)) := f \text{ and } \varphi_\Delta^{(3.3)}((f,a,g)) := g \text{ for } (f,a,g) \in \Delta^{(3)'},$$
$$\varphi_\Delta^{(3.1)}(f) := \varphi_\Delta^{(3.3)}(f) := f \text{ for } f \in S_\Delta^{(3)} \text{ and}$$
$$\varphi_\Delta^{(3.1)}(\check{0}) := \varphi_\Delta^{(3.3)}(\check{0}) := 0. \tag{120}$$

it holds $\gamma_\Delta^{(3)}(\varphi_\Delta^{(3)}(W_\Delta)) \subset E_{\check{\mathbb{P}}} \cap \Delta^{(3)\prime *}$ if

$$\begin{aligned}
&\varphi_\Delta^{(3)}(W_\Delta) \subset (\{\varepsilon\} \cup ((\varphi_\Delta^{(3.1)})^{-1}(0) \setminus \{\check{0}\})\Delta^{(3)*}) \ \setminus \ \Delta^{(3)*}F^{(3)}\Delta^{(3)*} \text{ where} \\
&F^{(3)} := \{xy \in \Delta^{(3)}\Delta^{(3)} | \varphi_\Delta^{(3.3)}(x) \neq \varphi_\Delta^{(3.1)}(y)\} \cup \\
&((\Delta^{(3)\prime} \cap (\varphi_\Delta^{(3.3)})^{-1}(0)) \cup \{\check{0}\})(\Delta^{(3)} \setminus \{\check{0}\}) \cup \\
&(\Delta^{(3)} \setminus ((\Delta^{(3)\prime} \cap (\varphi_\Delta^{(3.3)})^{-1}(0)) \cup \{\check{0}\}))\{\check{0}\}. \qquad (121)
\end{aligned}$$

Let the mapping $Z_\Delta^{(3)} : \Delta^{(3)*} \to S_\Delta^{(3)}$ be defined by

$$Z_\Delta^{(3)}(\varepsilon) := 0, \text{ and } Z_\Delta^{(3)}(uv) := \varphi_\Delta^{(3.3)}(v) \text{ for } u \in \Delta^{(3)*} \text{ and } v \in \Delta^{(3)}. \qquad (122)$$

Then (121) implies

$$Z_\Delta^{(3)}(\varphi_\Delta^{(3)}(x)) = Z_{\mathbb{P}}(\gamma_\Delta^{(3)}(\varphi_\Delta^{(3)}(x))) \text{ for each } x \in W_\Delta. \qquad (123)$$

Now the conditions (64a) and (64d) imply restrictions of the set $\Delta^{()\prime}$, which finally define the alphabet

$$\Delta^{()} \subset \Delta^{()\prime} = \Delta^{(1)} \times \Delta^{(2)} \times \Delta^{(3)} = \Delta \times (\Delta^{(2)\prime} \cup\!\!\!\cdot\, S_\Delta^{(2)}) \times (\Delta^{(3)\prime} \cup\!\!\!\cdot\, S_\Delta^{(3)} \cup\!\!\!\cdot\, \{\check{0}\}).$$

For that purpose let the mappings $\varphi_\Delta^{(2.2)} : \Delta^{(2)\prime} \to \Sigma$ and $\varphi_\Delta^{(3.2)} : \Delta^{(3)\prime} \to \check{\Sigma}$ be defined by

$$\varphi_\Delta^{(i.2)}((f,a,g)) := a \text{ for } (f,a,g) \in \Delta^{(i)\prime} \text{ with } i \in \{2,3\}. \qquad (124)$$

As the second and third track together represent a shuffled representation, (64a) requires

$$\begin{aligned}
&\text{either } x_2 \in \Delta^{(2)\prime},\ x_3 \in S_\Delta^{(3)} \cup\!\!\!\cdot\, \{\check{0}\} \text{ and } \varphi_\Delta^{(1.2)}(x_1) = \varphi_\Delta^{(2.2)}(x_2) \\
&\text{or } x_2 \in S_\Delta^{(2)},\ x_3 \in \Delta^{(3)\prime} \text{ and } \varphi_\Delta^{(1.2)}(x_1) = \check{\imath}(\varphi_\Delta^{(3.2)}(x_3)) \\
&\text{for each } (x_1,x_2,x_3) \in \Delta^{()}. \qquad (125)
\end{aligned}$$

Additionally (64d) requires

$$\begin{aligned}
&\varphi_\Delta^{(1.1)}(x_1) = \varphi_\Delta^{(2.1)}(x_2) + \varphi_\Delta^{(3.1)}(x_3) \text{ and} \\
&\varphi_\Delta^{(1.3)}(x_1) = \varphi_\Delta^{(2.3)}(x_2) + \varphi_\Delta^{(3.3)}(x_3) \text{ for each } (x_1,x_2,x_3) \in \Delta^{()}. \qquad (126)
\end{aligned}$$

Let therefore

$$\Delta^{()} := \{(x_1,x_2,x_3) \in \Delta^{()\prime} | \text{ it holds (125) and (126)}\}, \qquad (127)$$

which is finite and can be effectively determined, if Δ is finite.

Combining (127) with (105), (115) and (121) result in

Definition 25. *Let $\varDelta \subset \sqcup_{\mathbb{P}}$, then*

$$
\begin{aligned}
W_{\varDelta} :=& \varDelta^{()*} \cap \\
& (\varphi_{\varDelta}^{(1)})^{-1}[(\{\varepsilon\} \cup (\varphi_{\varDelta}^{(1.1)})^{-1}(0)\varDelta^{(1)*}) \ \setminus \ \varDelta^{(1)*}F^{(1)}\varDelta^{(1)*}] \cap \\
& (\varphi_{\varDelta}^{(2)})^{-1}[(\{\varepsilon\} \cup (\varphi_{\varDelta}^{(2.1)})^{-1}(0)\varDelta^{(2)*}) \ \setminus \ \varDelta^{(2)*}F^{(2)}\varDelta^{(2)*}] \cap \\
& (\varphi_{\varDelta}^{(3)})^{-1}[(\{\varepsilon\} \cup ((\varphi_{\varDelta}^{(3.1)})^{-1}(0) \setminus \{\check{0}\})\varDelta^{(3)*}) \ \setminus \ \varDelta^{(3)*}F^{(3)}\varDelta^{(3)*}].
\end{aligned}
$$

By the well known closure properties of the class of regular languages [1] this representation shows that $W_{\varDelta}$ is regular, if $\varDelta$ is finite, and it is a prefix closed local language, because of $\varphi_{\varDelta}^{(i)}(\varDelta^{()'}) \subset \varDelta^{(i)}$ for each $i \in \{1,2,3\}$.

To show that $W_{\varDelta}$ represents the function $\mathcal{R}'_{\mathbb{P}|2^{A_{\mathbb{P}}} \cap \varDelta^*}$, we need an additional homomorphism $\eta_{\varDelta} : \varDelta^{()*} \to (\sqcup_{\mathbb{P}} \dot{\cup} \sqcup_{\check{\mathbb{P}}})^*$, defined by

$$
\begin{aligned}
&\eta_{\varDelta}((x_1,x_2,x_3)) := x_2 \text{ for } (x_1,x_2,x_3) \in \varDelta^{()} \text{ with } x_2 \in \sqcup_{\mathbb{P}} \\
&\text{and} \\
&\eta_{\varDelta}((x_1,x_2,x_3)) := x_3 \text{ for } (x_1,x_2,x_3) \in \varDelta^{()} \text{ with } x_3 \in \sqcup_{\check{\mathbb{P}}}. \qquad (128)
\end{aligned}
$$

By (128) $\eta_{\varDelta}$ is well defined, because

$$\varDelta^{()} = \{(x_1,x_2,x_3) \in \varDelta^{()} | x_2 \in \sqcup_{\mathbb{P}}\} \dot{\cup} \{(x_1,x_2,x_3) \in \varDelta^{()} | x_3 \in \sqcup_{\check{\mathbb{P}}}\}$$

on account of (125).

(115) and (121) imply $\eta_{\varDelta}(W_{\varDelta}) \subset \pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}})$. With a standard induction technique for prefix closed local languages it follows

Lemma 9.
Let $x \in W_{\varDelta}$, then $\eta_{\varDelta}(x) \in \pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}})$ is a shuffled representation of $\varphi_{\varDelta}^{(1)}(x) \in A_{\mathbb{P}}$ by $\gamma_{\varDelta}^{(2)}(\varphi_{\varDelta}^{(2)}(x)) \in A_{\mathbb{P}}$ and . $\gamma_{\varDelta}^{(3)}(\varphi_{\varDelta}^{(3)}(x)) \in E_{\check{\mathbb{P}}}$.

To show the reverse of Lemma 9, the following observation is helpful:

Lemma 10.
Let $b', x \in (\sqcup_{\mathbb{P}} \dot{\cup} \sqcup_{\check{\mathbb{P}}})^*$ and $b = b'x \in \pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}})$ be a shuffled representation of $c \in A_{\mathbb{P}}$ by $d = \pi_{\sqcup_{\mathbb{P}}}(b) \in A_{\mathbb{P}}$ and $e = \pi_{\sqcup_{\check{\mathbb{P}}}}(b) \in E_{\check{\mathbb{P}}}$, then b' is a shuffled representation of $c' \in \mathrm{pre}(c)$ with $|c'| = |b'|$ by $d' \in \mathrm{pre}(d)$ and $e' \in \mathrm{pre}(e)$ with $|d'| = |\pi_{\sqcup_{\mathbb{P}}}(b')|$ and $|e'| = |\pi_{\sqcup_{\check{\mathbb{P}}}}(b')|$.

Using Lemma 10 with $|x| = 1$, standard induction technique shows

Lemma 11.
Let $b \in \pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}}) \cap \pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}})$ be a shuffled representation of $c \in A_{\mathbb{P}}$ by $d = \pi_{\sqcup_{\mathbb{P}}}(b) \in A_{\mathbb{P}}$ and $e = \pi_{\sqcup_{\check{\mathbb{P}}}}(b) \in E_{\check{\mathbb{P}}}$, then there exists $x \in W_{\varDelta}$ such that $b = \eta_{\varDelta}(x)$, $c = \varphi_{\varDelta}^{(1)}(x)$, $d = \gamma_{\varDelta}^{(2)}(\varphi_{\varDelta}^{(2)}(x))$ and $e = \gamma_{\varDelta}^{(3)}(\varphi_{\varDelta}^{(3)}(x))$.

Lemma 9 and Lemma 11 imply that for each $c \in A_{\mathbb{P}} \cap \Delta^*$ it holds,

$$d \in \mathcal{R}'_{\mathbb{P}}(\{c\}) \text{ iff there exists } x \in W_\Delta \text{ with } c = \varphi^{(1)}_\Delta(x) \text{ and } d = \gamma^{(2)}_\Delta(\varphi^{(2)}_\Delta(x)).$$

Now the results of this section can be summarized:

Definition 26.
Let the alphabetic homomorphisms $\mu_\Delta : \Delta^{()*} \to \Delta^*$ and $\nu_\Delta : \Delta^{()*} \to \sqcup^*_{\mathbb{P}}$ be defined by

$$\mu_\Delta(x) := \varphi^{(1)}_\Delta(x) \in \Delta^{(1)*} = \Delta^* \text{ and } \nu_\Delta(x) := \gamma^{(2)}_\Delta(\varphi^{(2)}_\Delta(x)) \in \Delta^{(2)'*} \subset \sqcup^*_{\mathbb{P}}$$

for $x \in \Delta^{()*} \subset \Delta^{()'*}$.

Theorem 16 (Representation Theorem).
Let $\Delta \subset \sqcup_{\mathbb{P}}$, then $\mathcal{R}'_{\mathbb{P}}(B) = \nu_\Delta(\mu^{-1}_\Delta(B) \cap W_\Delta)$ for each $B \subset A_{\mathbb{P}} \cap \Delta^*$. Additionally W_Δ is regular, if Δ is finite.

Example 13.
Theorem 16 can be applied to Example 12 to prove $\mathrm{SP}(\mathrm{pre}(\mathring{P}), \mathring{V})$. For that purpose a finite subset $\Delta \subset \sqcup_{\mathring{\mathbb{P}}}$ has to be found such that $\alpha^{-1}_{\mathring{\mathbb{P}}}(\mathring{V}) \subset A_{\mathring{\mathbb{P}}} \cap \Delta^*$. This can be achieved considering the product automaton of $\mathring{\mathbb{V}}$ and $\mathring{\mathbb{P}}_\sqcup$, if this automaton is finite. Reachability analysis for this product construction result in the product automaton of Fig. 12. Fig. 12 shows that

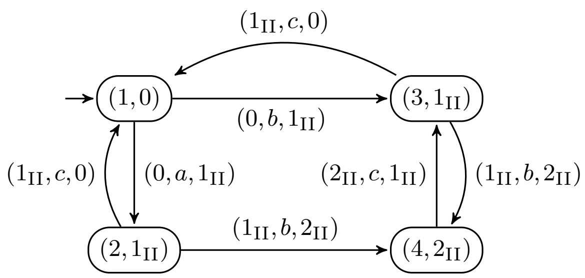


Fig. 12. Product automaton of $\mathring{\mathbb{V}}$ and $\mathring{\mathbb{P}}_\sqcup$

$$\alpha^{-1}_{\mathring{\mathbb{P}}}(\mathring{V}) \subset A_{\mathring{\mathbb{P}}} \cap \{(0, a, 1_{\mathrm{II}}), (0, b, 1_{\mathrm{II}}), (1_{\mathrm{II}}, c, 0), (1_{\mathrm{II}}, b, 2_{\mathrm{II}}), (2_{\mathrm{II}}, c, 1_{\mathrm{II}})\}^*. \quad (129)$$

For this example let therefore

$$\Delta := \Delta^{(1)} := \{(0, a, 1_{\mathrm{II}}), (0, b, 1_{\mathrm{II}}), (1_{\mathrm{II}}, c, 0), (1_{\mathrm{II}}, b, 2_{\mathrm{II}}), (2_{\mathrm{II}}, c, 1_{\mathrm{II}})\}.$$

This implies

$$S^{(1)}_\Delta = \{0, 1_{\mathrm{II}}, 2_{\mathrm{II}}\},\ S^{(3)}_\Delta = \{0, 1_{\mathrm{II}}\},\ S^{(2)}_\Delta = \{0, 1_{\mathrm{II}}, 2_{\mathrm{II}}\},\ \Sigma_\Delta = \{a, b, c\},$$

$$\Delta^{(2)'} = \{(0,a,1_{\mathrm{II}}),(0,b,1_{\mathrm{II}}),(1_{\mathrm{II}},c,0),(1_{\mathrm{II}},a,2_{\mathrm{II}}),(1_{\mathrm{II}},b,2_{\mathrm{II}}),(2_{\mathrm{II}},c,1_{\mathrm{II}})\}, \text{ and}$$

$$\Delta^{(3)'} = \{(0,a,1_{\mathrm{II}}),(0,b,1_{\mathrm{II}}),(1_{\mathrm{II}},c,0)\}.$$

Now $\Delta^{()}$ is given by (127). To illustrate the three tracks, we use a column notation to represent the elements of

$$\Delta^{()} = \{ \begin{bmatrix}(0,a,1_{\mathrm{II}})\\(0,a,1_{\mathrm{II}})\\0\end{bmatrix}, \begin{bmatrix}(0,a,1_{\mathrm{II}})\\0\\(0,\check{a},1_{\mathrm{II}})\end{bmatrix}, \begin{bmatrix}(0,b,1_{\mathrm{II}})\\(0,b,1_{\mathrm{II}})\\0\end{bmatrix}, \begin{bmatrix}(0,b,1_{\mathrm{II}})\\0\\(0,\check{b},1_{\mathrm{II}})\end{bmatrix}, \begin{bmatrix}(1_{\mathrm{II}},c,0)\\(1_{\mathrm{II}},c,0)\\0\end{bmatrix},$$

$$\begin{bmatrix}(1_{\mathrm{II}},c,0)\\0\\(1_{\mathrm{II}},\check{c},0)\end{bmatrix}, \begin{bmatrix}(1_{\mathrm{II}},b,2_{\mathrm{II}})\\(1_{\mathrm{II}},b,2_{\mathrm{II}})\\0\end{bmatrix}, \begin{bmatrix}(1_{\mathrm{II}},b,2_{\mathrm{II}})\\(0,b,1_{\mathrm{II}})\\1_{\mathrm{II}}\end{bmatrix}, \begin{bmatrix}(1_{\mathrm{II}},b,2_{\mathrm{II}})\\1_{\mathrm{II}}\\(0,\check{b},1_{\mathrm{II}})\end{bmatrix}, \begin{bmatrix}(2_{\mathrm{II}},c,1_{\mathrm{II}})\\(2_{\mathrm{II}},c,1_{\mathrm{II}})\\0\end{bmatrix},$$

$$\begin{bmatrix}(2_{\mathrm{II}},c,1_{\mathrm{II}})\\(1_{\mathrm{II}},c,0)\\1_{\mathrm{II}}\end{bmatrix}, \begin{bmatrix}(2_{\mathrm{II}},c,1_{\mathrm{II}})\\1_{\mathrm{II}}\\(1_{\mathrm{II}},\check{c},0)\end{bmatrix}, \begin{bmatrix}(0,a,1_{\mathrm{II}})\\(0,a,1_{\mathrm{II}})\\\check{0}\end{bmatrix}, \begin{bmatrix}(0,b,1_{\mathrm{II}})\\(0,b,1_{\mathrm{II}})\\\check{0}\end{bmatrix}, \begin{bmatrix}(1_{\mathrm{II}},c,0)\\(1_{\mathrm{II}},c,0)\\\check{0}\end{bmatrix},$$

$$\begin{bmatrix}(1_{\mathrm{II}},b,2_{\mathrm{II}})\\(1_{\mathrm{II}},b,2_{\mathrm{II}})\\\check{0}\end{bmatrix}, \begin{bmatrix}(2_{\mathrm{II}},c,1_{\mathrm{II}})\\(2_{\mathrm{II}},c,1_{\mathrm{II}})\\\check{0}\end{bmatrix}\}.$$

The definition of W_Δ can be translated into a semiautomaton

$$\mathbb{W}_\Delta := (\Delta^{()} \ , \ S_\Delta^{()} \ , \ \Lambda_\Delta \ , \ (0,0,0))$$

recognizing W_Δ, where $S_\Delta^{()} := S_\Delta^{(1)} \times S_\Delta^{(2)} \times (S_\Delta^{(3)} \cup \{\check{0}\})$. Its state transition relation

$$\Lambda_\Delta \subset S_\Delta^{()} \times \Delta^{()} \times S_\Delta^{()}$$

can be constructed step by step in compliance with the restrictions of Definition 25. For its representation we use a column notation for the states just as for the elements of $\Delta^{()}$. So we get

$$\Lambda_\Delta = \{ \begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}(0,a,1_{\mathrm{II}})\\(0,a,1_{\mathrm{II}})\\0\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}(0,a,1_{\mathrm{II}})\\0\\(0,\check{a},1_{\mathrm{II}})\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\0\\1_{\mathrm{II}}\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}(0,b,1_{\mathrm{II}})\\(0,b,1_{\mathrm{II}})\\0\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}(0,b,1_{\mathrm{II}})\\0\\(0,\check{b},1_{\mathrm{II}})\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\0\\1_{\mathrm{II}}\end{bmatrix}, \begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\0\end{bmatrix}\begin{bmatrix}(1_{\mathrm{II}},c,0)\\(1_{\mathrm{II}},c,0)\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\0\end{bmatrix}\begin{bmatrix}(1_{\mathrm{II}},b,2_{\mathrm{II}})\\(1_{\mathrm{II}},b,2_{\mathrm{II}})\\0\end{bmatrix}\begin{bmatrix}2_{\mathrm{II}}\\2_{\mathrm{II}}\\0\end{bmatrix},$$

$$\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\0\end{bmatrix}\begin{bmatrix}(1_{\mathrm{II}},b,2_{\mathrm{II}})\\1_{\mathrm{II}}\\(0,\check{b},1_{\mathrm{II}})\end{bmatrix}\begin{bmatrix}2_{\mathrm{II}}\\1_{\mathrm{II}}\\1_{\mathrm{II}}\end{bmatrix}, \begin{bmatrix}1_{\mathrm{II}}\\0\\1_{\mathrm{II}}\end{bmatrix}\begin{bmatrix}(1_{\mathrm{II}},c,0)\\0\\(1_{\mathrm{II}},\check{c},0)\end{bmatrix}\begin{bmatrix}0\\0\\\check{0}\end{bmatrix}, \begin{bmatrix}1_{\mathrm{II}}\\0\\1_{\mathrm{II}}\end{bmatrix}\begin{bmatrix}(1_{\mathrm{II}},b,2_{\mathrm{II}})\\(0,b,1_{\mathrm{II}})\\1_{\mathrm{II}}\end{bmatrix}\begin{bmatrix}2_{\mathrm{II}}\\1_{\mathrm{II}}\\1_{\mathrm{II}}\end{bmatrix},$$

$$\begin{bmatrix}2_{\mathrm{II}}\\2_{\mathrm{II}}\\0\end{bmatrix}\begin{bmatrix}(2_{\mathrm{II}},c,1_{\mathrm{II}})\\(2_{\mathrm{II}},c,1_{\mathrm{II}})\\0\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\0\end{bmatrix},\begin{bmatrix}2_{\mathrm{II}}\\1_{\mathrm{II}}\\1_{\mathrm{II}}\end{bmatrix}\begin{bmatrix}(2_{\mathrm{II}},c,1_{\mathrm{II}})\\(1_{\mathrm{II}},c,0)\\1_{\mathrm{II}}\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\0\\1_{\mathrm{II}}\end{bmatrix},\begin{bmatrix}2_{\mathrm{II}}\\1_{\mathrm{II}}\\1_{\mathrm{II}}\end{bmatrix}\begin{bmatrix}(2_{\mathrm{II}},c,1_{\mathrm{II}})\\1_{\mathrm{II}}\\(1_{\mathrm{II}},\check{c},0)\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\\check{0}\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\\check{0}\end{bmatrix}\begin{bmatrix}(0,a,1_{\mathrm{II}})\\(0,a,1_{\mathrm{II}})\\\check{0}\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\\check{0}\end{bmatrix},\begin{bmatrix}0\\0\\\check{0}\end{bmatrix}\begin{bmatrix}(0,b,1_{\mathrm{II}})\\(0,b,1_{\mathrm{II}})\\\check{0}\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\\check{0}\end{bmatrix},\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\\check{0}\end{bmatrix}\begin{bmatrix}(1_{\mathrm{II}},c,0)\\(1_{\mathrm{II}},c,0)\\\check{0}\end{bmatrix}\begin{bmatrix}0\\0\\\check{0}\end{bmatrix},$$

$$\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\\check{0}\end{bmatrix}\begin{bmatrix}(1_{\mathrm{II}},b,2_{\mathrm{II}})\\(1_{\mathrm{II}},b,2_{\mathrm{II}})\\\check{0}\end{bmatrix}\begin{bmatrix}2_{\mathrm{II}}\\2_{\mathrm{II}}\\\check{0}\end{bmatrix},\begin{bmatrix}2_{\mathrm{II}}\\2_{\mathrm{II}}\\\check{0}\end{bmatrix}\begin{bmatrix}(2_{\mathrm{II}},c,1_{\mathrm{II}})\\(2_{\mathrm{II}},c,1_{\mathrm{II}})\\\check{0}\end{bmatrix}\begin{bmatrix}1_{\mathrm{II}}\\1_{\mathrm{II}}\\\check{0}\end{bmatrix}\}.$$

Applying standard automata algorithms [1] to this semiautomaton, shows $\nu_\Delta(W_\Delta) \subset \alpha_{\mathring{\mathbb{P}}}^{-1}(\mathring{V})$, which by Theorem 16 and (129) implies

$$\mathcal{R}'_{\mathring{\mathbb{P}}}(\alpha_{\mathring{\mathbb{P}}}^{-1}(\mathring{V})) = \nu_\Delta(\mu_\Delta^{-1}(\alpha_{\mathring{\mathbb{P}}}^{-1}(\mathring{V})) \cap W_\Delta) \subset \alpha_{\mathring{\mathbb{P}}}^{-1}(\mathring{V}). \tag{130}$$

Now (130) together with Corollary 8 proves $\mathrm{SP}(\mathrm{pre}(\mathring{P}), \mathring{V})$.

Using Corollary 8 and Theorem 16, Example 13 demonstrates how to decide $\mathrm{SP}(\mathrm{pre}(P), V)$, if there exists a finite subset $\Delta \subset \sqcup\!\sqcup_{\mathbb{P}}$, such that $\alpha_{\mathbb{P}}^{-1}(V) \subset \Delta^*$. Since we assume $\emptyset \neq P \subset \Sigma^*$ and $\delta(q_0, \mathrm{pre}(P)) = Q$, $\mathrm{pre}(P)$ is recognized by the automaton $\grave{\mathbb{P}} := (\Sigma, Q, \delta, q_0, Q)$. So using Corollary 9 instead of Corollary 8, we also can decide $\mathrm{SP}(\mathrm{pre}(P), V)$, if there exists a finite subset $\grave{\Delta} \subset \sqcup\!\sqcup_{\grave{\mathbb{P}}}$ such that $(\alpha_{\grave{\mathbb{P}}}^{-1}(V) \cap Z_{\grave{\mathbb{P}}}^{-1}(0)) \subset \grave{\Delta}^*$.

Now the question arises: Is there any relation between Δ and $\grave{\Delta}$? The only difference between $\mathbb{P}$ and $\grave{\mathbb{P}}$ is the set of their final states: $F \subset Q$ versus Q. Therefore Definition 16 implies

$$\begin{aligned}\sqcup\!\sqcup_{\grave{\mathbb{P}}} = \sqcup\!\sqcup_{\mathbb{P}} \;\cup\; &\{(f,a,f) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0,a) \text{ is defined}\} \cup \\ &\{(f,a,f-1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q \text{ and } \delta(q,a) \text{ is defined}\}.\end{aligned}$$

Now on account of

$$\begin{aligned}&\{(f,a,f-1_q+1_{\delta(q,a)}) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q \text{ and } \delta(q,a) \text{ is defined}\} \cup \\ &\{(f,a,f+1_{\delta(q_0,a)}) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0,a) \text{ is defined}\} \subset \sqcup\!\sqcup_{\mathbb{P}},\end{aligned}$$

$$\text{for each } (f,a,g) \in \sqcup\!\sqcup_{\grave{\mathbb{P}}} \text{ there exists } (f,a,g') \in \sqcup\!\sqcup_{\mathbb{P}} \text{ such that } g' \geqslant g. \tag{131}$$

(35) implies

$$(f+h, a, g'+h) \in \sqcup\!\sqcup_{\mathbb{P}} \text{ for each } (f,a,g') \in \sqcup\!\sqcup_{\mathbb{P}} \text{ and } h \in \mathbb{N}_0^Q. \tag{132}$$

By (131) and (132) an induction proof shows:

For each $x \in A_{\grave{\mathbb{P}}}$ there exists $y \in A_{\mathbb{P}}$ with $\alpha_{\grave{\mathbb{P}}}(x) = \alpha_{\mathbb{P}}(y)$ and
$Z_{\mathbb{P}}(y') \geqslant Z_{\grave{\mathbb{P}}}(x')$ for each $x' \in \mathrm{pre}(x)$ and $y' \in \mathrm{pre}(y)$ with $|x'| = |y'|$,

which implies

$$\begin{aligned}&\text{For each } x \in \alpha_{\grave{\mathbb{P}}}^{-1}(V) \text{ there exists } y \in \alpha_{\mathbb{P}}^{-1}(V) \text{ with } \alpha_{\grave{\mathbb{P}}}(x) = \alpha_{\mathbb{P}}(y) \text{ and}\\ &Z_{\mathbb{P}}(y') \geqslant Z_{\grave{\mathbb{P}}}(x') \text{ for each } x' \in \mathrm{pre}(x) \text{ and } y' \in \mathrm{pre}(y) \text{ with } |x'| = |y'|.\end{aligned} \tag{133}$$

Let now $\Delta \subset \sqcup\!\sqcup_{\mathbb{P}} \cap (T(Q) \times \Sigma \times T(Q))$ such that $\alpha_{\mathbb{P}}^{-1}(V) \subset \Delta^*$, and let Σ_Δ be defined as in (110). Let

$$\begin{aligned}&S_\Delta := Z_{\mathbb{P}}(\mathrm{pre}(\alpha_{\mathbb{P}}^{-1}(V))) \text{ and}\\ &\grave{S}_\Delta := \{f \in \mathbb{N}_0^Q \mid \text{there exists } g \in S_\Delta \text{ with } g \geqslant f\}.\end{aligned} \tag{134}$$

Then finiteness of Δ implies finiteness of Σ_Δ, S_Δ and $\grave{S}_\Delta$, and by (133) holds $\alpha_{\grave{\mathbb{P}}}^{-1}(V) \subset \grave{\Delta}^*$ with $\grave{\Delta} := \sqcup\!\sqcup_{\grave{\mathbb{P}}} \cap (\grave{S}_\Delta \times \Sigma_\Delta \times \grave{S}_\Delta)$. This implies:

$$\begin{aligned}&\text{If } \alpha_{\mathbb{P}}^{-1}(V) \subset \Delta^* \text{ for a finite subset } \Delta \subset \sqcup\!\sqcup_{\mathbb{P}}, \text{ then there exists}\\ &\text{a finite subset } \grave{\Delta} \subset \sqcup\!\sqcup_{\grave{\mathbb{P}}} \text{ with } (\alpha_{\grave{\mathbb{P}}}^{-1}(V) \cap Z_{\grave{\mathbb{P}}}^{-1}(0)) \subset \grave{\Delta}^*.\end{aligned} \tag{135}$$

The following example shows that the converse of (135) does not hold.

Example 14.
Let $\bar{\mathbb{P}}$ and $\bar{P}$ as defined in Figure 8, and let $\bar{\mathbb{V}}$ and $\bar{V}$ as defined in Figure 13. Then $Z_{\bar{\mathbb{P}}}(\alpha_{\bar{\mathbb{P}}}^{-1}(\bar{V})) = \{0\} \cup \{n_{\mathrm{II}} | n \in \mathbb{N}\}$. Therefore each $\Delta \subset \sqcup\!\sqcup_{\bar{\mathbb{P}}}$ with $\alpha_{\bar{\mathbb{P}}}^{-1}(\bar{V}) \subset \Delta^*$ is an infinite set.

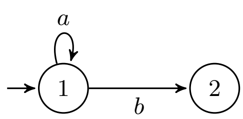


Fig. 13. Semiautomaton $\bar{\mathbb{V}}$ recognizing $\bar{V}$

But $(\alpha_{\grave{\bar{\mathbb{P}}}}^{-1}(\bar{V}) \cap Z_{\grave{\bar{\mathbb{P}}}}^{-1}(0)) \subset \{(0,a,0),(0,a,1_{\mathrm{II}}),(1_{\mathrm{II}},a,1_{\mathrm{II}}),(1_{\mathrm{II}},b,0)\}^*$, because of $Z_{\grave{\bar{\mathbb{P}}}}(\alpha_{\grave{\bar{\mathbb{P}}}}^{-1}(\bar{V})) = \{0\} \cup \{n_{\mathrm{II}} | n \in \mathbb{N}\}$, and $c^{-1}(\alpha_{\grave{\bar{\mathbb{P}}}}^{-1}(\bar{V}) \cap Z_{\grave{\bar{\mathbb{P}}}}^{-1}(0)) = \emptyset$ for each $c \in \alpha_{\grave{\bar{\mathbb{P}}}}^{-1}(\bar{V})$ with $Z_{\grave{\bar{\mathbb{P}}}}(c) \geqslant 2_{\mathrm{II}}$.

8 Decidability Questions

In Section 7 it was demonstrated by way of an example, how a finite set $\Delta \subset \sqcup_{\mathbb{P}}$ can be found that fulfils the condition $\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \subset \Delta^*$. Given that P and V are regular languages the approach is now considered in general and it is shown how the existence of such a finite set can be decided.

For an arbitrary alphabet Γ let the mapping $\mathrm{alph} : 2^{\Gamma^*} \to 2^{\Gamma}$ be defined by $\Gamma(\emptyset) := \Gamma(\{\varepsilon\}) := \emptyset$, $\Gamma(\{wa\}) := \Gamma(\{w\}) \cup \{a\}$ for $w \in \Gamma^*$ and $a \in \Gamma$, and $\Gamma(L) := \bigcup_{w \in L} \Gamma(\{w\})$. Then the minimal set Δ with the above property is $\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))$. So, the problem is to find $\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))$ and to prove that $\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))$ is finite. In (135) it is shown that the more general problem is to investigate $\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0))$. But we first examine the problem concerning $\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))$, because there is a much easier decision procedure than for the general problem.

Let now $\emptyset \neq P \subset \Sigma^*$, $\emptyset \neq V \subset \Sigma^*$, $\mathbb{P} = (\Sigma, Q, \delta, q_0, F)$ a deterministic automaton that recognizes P with $\delta(q_0, \mathrm{pre}(P)) = Q$, and $\mathbb{V} = (\Sigma, Q_{\mathbb{V}}, \delta_{\mathbb{V}}, q_{\mathbb{V}_0})$ a deterministic semiautomaton that recognizes $\mathrm{pre}(V)$ with $Q \cap Q_{\mathbb{V}} = \emptyset$. Then $\delta_{\mathbb{V}}(q_{\mathbb{V}_0}, \alpha_{\mathbb{P}}(x))$ is defined for each $x \in \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))$.

$$\begin{aligned}&\text{The set } x^{-1}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \cap \sqcup_{\mathbb{P}} \text{ is finite for each } x \in \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \\ &\qquad \text{and depends only on } (Z_{\mathbb{P}}(x), \delta_{\mathbb{V}}(q_{\mathbb{V}_0}, \alpha_{\mathbb{P}}(x))).\end{aligned} \tag{136}$$

$$\begin{aligned}&\text{For each } y \in x^{-1}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \cap \sqcup_{\mathbb{P}} \text{ is } (Z_{\mathbb{P}}(xy), \delta_{\mathbb{V}}(q_{\mathbb{V}_0}, \alpha_{\mathbb{P}}(xy))) \\ &\qquad \text{uniquely determined by } (Z_{\mathbb{P}}(x), \delta_{\mathbb{V}}(q_{\mathbb{V}_0}, \alpha_{\mathbb{P}}(x))) \text{ and } y.\end{aligned} \tag{137}$$

Let $Q_{\mathbb{PV}} := \{(Z_{\mathbb{P}}(x), \delta_{\mathbb{V}}(q_{\mathbb{V}_0}, \alpha_{\mathbb{P}}(x))) | x \in \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))\}$. Then $Q_{\mathbb{PV}}$ can be considered as the state set of a deterministic semiautomaton $\mathbb{S}_{\mathbb{PV}}$ that recognizes $\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))$. Its initial state is $(0, q_{\mathbb{V}_0})$, its alphabet is $\sqcup_{\mathbb{P}}$, and its state transition function is given by (137). More precisely:

$$\begin{aligned}&\mathbb{S}_{\mathbb{PV}} = (\sqcup_{\mathbb{P}}, Q_{\mathbb{PV}}, \delta_{\mathbb{PV}}, (0, q_{\mathbb{V}_0})) \text{ where } \delta_{\mathbb{PV}} : Q_{\mathbb{PV}} \times \sqcup_{\mathbb{P}} \to Q_{\mathbb{PV}} \\ &\text{is a partial function with} \\ &\delta_{\mathbb{PV}}((Z_{\mathbb{P}}(x), \delta_{\mathbb{V}}(q_{\mathbb{V}_0}, \alpha_{\mathbb{P}}(x))), y) := (Z_{\mathbb{P}}(xy), \delta_{\mathbb{V}}(q_{\mathbb{V}_0}, \alpha_{\mathbb{P}}(xy))) \\ &\text{for } x \in \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \text{ and } y \in x^{-1}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \cap \sqcup_{\mathbb{P}}.\end{aligned} \tag{138}$$

In example 13, $\mathbb{S}_{\mathbb{PV}}$ corresponds to the product automaton of Figure 12.

Let now $Z_{\mathbb{PV}} : \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \to Q_{\mathbb{PV}}$ with

$$Z_{\mathbb{PV}}(x) := (Z_{\mathbb{P}}(x), \delta_{\mathbb{V}}(q_{\mathbb{V}_0}, \alpha_{\mathbb{P}}(x))) \text{ for each } x \in \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)). \tag{139}$$

Then

$$Q_{\mathbb{PV}} = Z_{\mathbb{PV}}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \text{ and}$$
$$Z_{\mathbb{PV}}(x) = \delta_{\mathbb{PV}}((0, q_{\mathbb{V}0}), x) \text{ for each } x \in \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)). \tag{140}$$

$$\text{For each } n \in \mathbb{N}_0 \text{ let } A_{\mathbb{P}}^{(n)} := \{w \in A_{\mathbb{P}} \,|\, |w| \leq n\} \text{ and}$$
$$Q_{\mathbb{PV}}^{(n)} := Z_{\mathbb{PV}}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \cap A_{\mathbb{P}}^{(n)}). \tag{141}$$

$$\text{From (136) follows that } \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \cap A_{\mathbb{P}}^{(n)} \text{ and thus}$$
$$Q_{\mathbb{PV}}^{(n)} \text{ for each } n \in \mathbb{N}_0 \text{ are finite sets.} \tag{142}$$

If $Q_{\mathbb{PV}}^{(k)} = Q_{\mathbb{PV}}^{(k+1)}$ for a $k \in \mathbb{N}_0$, then follows from (136) and (137) $Q_{\mathbb{PV}}^{(i)} = Q_{\mathbb{PV}}^{(k)}$ and

$$\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \cap A_{\mathbb{P}}^{(i+1)}) = \mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \cap A_{\mathbb{P}}^{(k+1)}) \text{ for each } i \geq k. \tag{143}$$

Because $A_{\mathbb{P}} = \bigcup_{n \in \mathbb{N}_0} A_{\mathbb{P}}^{(n)}$ and $A_{\mathbb{P}}^{(n)} \subset A_{\mathbb{P}}^{(n+1)}$ for each $n \in \mathbb{N}_0$ holds

$$Q_{\mathbb{PV}} = \bigcup_{n \in \mathbb{N}_0} Q_{\mathbb{PV}}^{(n)} \text{ and } Q_{\mathbb{PV}}^{(n)} \subset Q_{\mathbb{PV}}^{(n+1)} \text{ for each } n \in \mathbb{N}_0. \tag{144}$$

From (143)-(144) follows

$$\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) = \mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \cap A_{\mathbb{P}}^{(k+1)}), \text{ as well as } Q_{\mathbb{PV}} = Q_{\mathbb{PV}}^{(k)} \text{ if}$$
$$Q_{\mathbb{PV}}^{(k)} = Q_{\mathbb{PV}}^{(k+1)}, \text{ and } \mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \text{ and } Q_{\mathbb{PV}} \text{ are finite sets} \tag{145}$$

because of (142).

Because $\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))$ is prefix closed

$$\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \subset Z_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \times \Sigma \times Z_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))), \text{ and}$$

$Z_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \subset p_3(\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))) \cup \{0\}$, where $p_3((f,a,g)) := g$ for $(f,a,g) \in \sqcup_{\mathbb{P}}$. Because Σ is finite, it follows

$$\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \text{ is finite iff } Z_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \text{ is finite.} \tag{146}$$

Accordingly, from the finiteness of $Q_{\mathbb{V}}$ follows

$$Z_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))) \text{ is finite iff } Q_{\mathbb{PV}} \text{ is finite.} \tag{147}$$

If $Q_{\mathbb{PV}}$ is finite, then because of (144)

$$\text{it exists a } k \in \mathbb{N}_0 \text{ with } Q_{\mathbb{PV}}^{(i)} = Q_{\mathbb{PV}}^{(k)} \text{ for all } i \geq k. \tag{148}$$

Because of (145)-(148) the stepwise computation of

$$Q_{\mathbb{PV}}^{(i)} \text{ for } i \in \mathbb{N}_0 \text{ and the test } Q_{\mathbb{PV}}^{(i)} = Q_{\mathbb{PV}}^{(i+1)}$$
$$\text{provides a semi-algorithm for the finiteness of } \mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))). \tag{149}$$

In case of a positive result, $\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))$ can be computed using (145).

In preparation for the decision on finiteness of $Q_{\mathbb{PV}}$ we need a closer look on the structure of $\shuffle_{\mathbb{P}}$. By Definition 13, Definition 16 and (50) it holds

$$\begin{aligned}
\shuffle_{\mathbb{P}} &= \wedge_{\mathbb{P}}(\hat{\shuffle}_{\mathbb{P}}) = \wedge_{\mathbb{P}}(\tilde{\shuffle}_{\mathbb{P}}) \cup \wedge_{\mathbb{P}}(\mathring{\shuffle}_{\mathbb{P}}) \cup \wedge_{\mathbb{P}}(\bar{\shuffle}_{\mathbb{P}}) \cup \wedge_{\mathbb{P}}(\tilde{\bar{\shuffle}}_{\mathbb{P}}) \text{ and} \\
\wedge_{\mathbb{P}}(\tilde{\shuffle}_{\mathbb{P}}) &= \{(f,a,f+1_p) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0,a) = p \text{ and it exists } b \in \Sigma \text{ such} \\
&\qquad \text{that } \delta(p,b) \text{ is defined}\}, \\
\wedge_{\mathbb{P}}(\mathring{\shuffle}_{\mathbb{P}}) &= \{(f,a,f+1_p-1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q, \delta(q,a) = p \text{ and it exists} \\
&\qquad b \in \Sigma \text{ such that } \delta(p,b) \text{ is defined}\}, \\
\wedge_{\mathbb{P}}(\bar{\shuffle}_{\mathbb{P}}) &= \{(f,a,f-1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q \text{ and } \delta(q,a) \in F\} \text{ and} \\
\wedge_{\mathbb{P}}(\tilde{\bar{\shuffle}}_{\mathbb{P}}) &= \{(f,a,f) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0,a) \in F\}.
\end{aligned} \tag{150}$$

On account of (132) a proper subset $\shuffle_{\mathbb{P}}^{\sigma} \subset \shuffle_{\mathbb{P}}$ together with $\mathbb{N}_0^Q$ suffices to completely characterize $\shuffle_{\mathbb{P}}$. Let therefore

$$\begin{aligned}
\shuffle_{\mathbb{P}}^{\sigma} &:= \tilde{\shuffle}_{\mathbb{P}}^{\sigma} \cup \mathring{\shuffle}_{\mathbb{P}}^{\sigma} \cup \bar{\shuffle}_{\mathbb{P}}^{\sigma} \cup \tilde{\bar{\shuffle}}_{\mathbb{P}}^{\sigma} \text{ with} \\
\tilde{\shuffle}_{\mathbb{P}}^{\sigma} &:= \{(0,a,1_p) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0,a) = p \text{ and it exists } b \in \Sigma \text{ such} \\
&\qquad \text{that } \delta(p,b) \text{ is defined}\}, \\
\mathring{\shuffle}_{\mathbb{P}}^{\sigma} &:= \{(1_q,a,1_p) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q,a) = p \text{ and it exists} \\
&\qquad b \in \Sigma \text{ such that } \delta(p,b) \text{ is defined}\}, \\
\bar{\shuffle}_{\mathbb{P}}^{\sigma} &:= \{(1_q,a,0) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q,a) \in F\} \text{ and} \\
\tilde{\bar{\shuffle}}_{\mathbb{P}}^{\sigma} &:= \{(0,a,0) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0,a) \in F\}.
\end{aligned} \tag{151}$$

Then by (132)

$$\shuffle_{\mathbb{P}} = \{(f+h,a,g+h) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid (f,a,g) \in \shuffle_{\mathbb{P}}^{\sigma} \text{ and } h \in \mathbb{N}_0^Q\}. \tag{152}$$

The following should be noticed:

$$\begin{aligned}
&\shuffle_{\mathbb{P}}^{\sigma} = \tilde{\shuffle}_{\mathbb{P}}^{\sigma} \uplus \mathring{\shuffle}_{\mathbb{P}}^{\sigma} \uplus \bar{\shuffle}_{\mathbb{P}}^{\sigma} \uplus \tilde{\bar{\shuffle}}_{\mathbb{P}}^{\sigma}. \\
&\text{Generally, for } (f',a,g') \in \shuffle_{\mathbb{P}} \text{ the representation } (f',a,g') = (f+h,a,g+h) \\
&\text{with } (f,a,g) \in \shuffle_{\mathbb{P}}^{\sigma} \text{ and } h \in \mathbb{N}_0^Q \text{ is not unique.} \\
&\shuffle_{\mathbb{P}}^{\sigma} \text{ is finite for finite automata } \mathbb{P}.
\end{aligned} \tag{153}$$

Let the mapping $\sigma_{\mathbb{P}} : \shuffle_{\mathbb{P}} \to 2^{\shuffle_{\mathbb{P}}^{\sigma}} \setminus \{\emptyset\}$ be defined by

$$\sigma_{\mathbb{P}}((f',a,g')) := \{(f,a,g) \in \shuffle_{\mathbb{P}}^{\sigma} \mid (f',a,g') = (f+h,a,g+h) \text{ with } h \in \mathbb{N}_0^Q\} \tag{154}$$

for $(f', a, g') \in \sqcup_{\mathbb{P}}$.

For the decision on finiteness of $Q_{\mathbb{P}\mathbb{V}}$ we now utilize results from Petri nets [12], [13]. A Petri net $N = (S, T, K)$ consists of a finite set S of *places*, a finite set T of *transitions*, and a set $K \subset (S \times T) \cup (T \times S)$ of *edges*. A *marking* of such a Petri net is a mapping $M : S \to \mathbb{N}_0$. Dynamic behavior of Petri nets is formalized in terms of *occurrence steps* and *occurrence sequences*. The set Ω of occurrence steps is defined by

$$\begin{aligned}\Omega := \{(M, t, M') \in \mathbb{N}_0^S \times T \times \mathbb{N}_0^S \mid M \geqslant \sum_{x \in S, (x,t) \in K} 1_x \text{ and} \\ M' = M - \sum_{x \in S, (x,t) \in K} 1_x + \sum_{y \in S, (t,y) \in K} 1_y\}.\end{aligned} \tag{155}$$

The set $\mathcal{O}$ of occurrence steps with $\mathcal{O} \subset \Omega^+$ and the functions $\mathcal{I} : \mathcal{O} \to \mathbb{N}_0^S$ and $\mathcal{F} : \mathcal{O} \to \mathbb{N}_0^S$ are defined inductively by

$$\begin{aligned}&\text{For each } o = (M, t, M') \in \Omega \text{ let } o \in \mathcal{O},\ \mathcal{I}(o) := M \text{ and } \mathcal{F}(o) := M'.\\ &\text{For each } w \in \mathcal{O} \text{ and } o \in \Omega \text{ with } \mathcal{F}(w) = \mathcal{I}(o) \text{ let}\\ &wo \in \mathcal{O},\ \mathcal{I}(wo) := \mathcal{I}(w) \text{ and } \mathcal{F}(wo) := \mathcal{F}(o).\end{aligned} \tag{156}$$

$\mathcal{I}(w)$ is called the *initial marking* and $\mathcal{F}(w)$ the *final marking* of w. For $M \in \mathbb{N}_0^S$ the *reachability set* $\mathcal{E}(M)$ is defined by

$$\mathcal{E}(M) := \{M\} \cup \mathcal{F}(\mathcal{I}^{-1}(M)). \tag{157}$$

The semiautomaton $\mathbb{S}_{\mathbb{P}\mathbb{V}}$ can be simulated by a Petri net $N_{\mathbb{P}\mathbb{V}}$ such that there exists an injective mapping ι from $Q_{\mathbb{P}\mathbb{V}}$ into the set of markings of $N_{\mathbb{P}\mathbb{V}}$ with

$$\iota(Q_{\mathbb{P}\mathbb{V}}) = \mathcal{E}(\iota((0, q_{\mathbb{V}_0}))). \tag{158}$$

To define $N_{\mathbb{P}\mathbb{V}}$ let its set of places $S := Q \dot\cup Q_{\mathbb{V}}$. Let therefore the injective mapping

$$\iota : Q_{\mathbb{P}\mathbb{V}} \to \mathbb{N}_0^{Q \dot\cup Q_{\mathbb{V}}}$$

be defined by

$$\begin{aligned}&\iota((f,q))(x) := f(x) \text{ for } x \in Q,\\ &\iota((f,q))(x) := 0 \text{ for } x \in Q_{\mathbb{V}} \setminus \{q\} \text{ and}\\ &\iota((f,q))(x) := 1 \text{ for } x \in Q_{\mathbb{V}} \cap \{q\},\\ &\text{for each } (f,q) \in Q_{\mathbb{P}\mathbb{V}} \subset \mathbb{N}_0^Q \times Q_{\mathbb{V}}.\end{aligned} \tag{159}$$

The set T of transitions of $N_{\mathbb{P}\mathbb{V}}$ will be defined such that there exists a bijective mapping $\chi : \sqcup_{\mathbb{P}}^{\sigma} \times Q_{\mathbb{V}} \to T$. For this purpose let $T := \tilde{T} \cup \mathring{T} \cup \bar{T} \cup \tilde{\bar{T}}$, where

$$\begin{aligned}\tilde{T} :=& \{(r, a, (p, s)) \in Q_{\mathbb{V}} \times \Sigma \times (Q \times Q_{\mathbb{V}}) \mid (0, a, 1_p) \in \tilde{\sqcup}_{\mathbb{P}}^{\sigma} \text{ and } \delta_{\mathbb{V}}(r, a) = s\},\\ \mathring{T} :=& \{((q, r), a, (p, s)) \in (Q \times Q_{\mathbb{V}}) \times \Sigma \times (Q \times Q_{\mathbb{V}}) \mid (1_q, a, 1_p) \in \mathring{\sqcup}_{\mathbb{P}}^{\sigma} \text{ and } \delta_{\mathbb{V}}(r, a) = s\},\\ \bar{T} :=& \{((q, r), a, s) \in (Q \times Q_{\mathbb{V}}) \times \Sigma \times Q_{\mathbb{V}} \mid (1_q, a, 0) \in \bar{\sqcup}_{\mathbb{P}}^{\sigma} \text{ and } \delta_{\mathbb{V}}(r, a) = s\}, \text{ and}\\ \tilde{\bar{T}} :=& \{(r, a, s) \in Q_{\mathbb{V}} \times \Sigma \times Q_{\mathbb{V}} \mid (0, a, 0) \in \tilde{\bar{\sqcup}}_{\mathbb{P}}^{\sigma} \text{ and } \delta_{\mathbb{V}}(r, a) = s\}.\end{aligned} \tag{160}$$

Now let the bijective mapping $\chi : \sqcup\!\sqcup_{\mathbb{P}}^{\sigma} \times Q_{\mathbb{V}} \to T$ be defined by

$$\begin{aligned}
\chi(((0,a,1_p),r)) &:= (r,a,(p,\delta_{\mathbb{V}}(r,a))) \text{ for } ((0,a,1_p),r) \in \tilde{\sqcup\!\sqcup}_{\mathbb{P}}^{\sigma} \times Q_{\mathbb{V}}, \\
\chi(((1_q,a,1_p),r)) &:= ((q,r),a,(p,\delta_{\mathbb{V}}(r,a))) \text{ for } ((1_q,a,1_p),r) \in \mathring{\sqcup\!\sqcup}_{\mathbb{P}}^{\sigma} \times Q_{\mathbb{V}}, \\
\chi(((1_q,a,0),r)) &:= ((q,r),a,\delta_{\mathbb{V}}(r,a)) \text{ for } ((1_q,a,0),r) \in \bar{\sqcup\!\sqcup}_{\mathbb{P}}^{\sigma} \times Q_{\mathbb{V}}, \text{ and} \\
\chi(((0,a,0),r)) &:= (r,a,\delta_{\mathbb{V}}(r,a)) \text{ for } ((0,a,0),r) \in \tilde{\bar{\sqcup\!\sqcup}}_{\mathbb{P}}^{\sigma} \times Q_{\mathbb{V}}.
\end{aligned} \tag{161}$$

The set K of edges of $N_{\mathbb{PV}}$ let be defined by $K := \tilde{K} \cup \mathring{K} \cup \bar{K} \cup \tilde{\bar{K}}$, where

$$\begin{aligned}
\tilde{K} &:= \bigcup_{(r,a,(p,s)) \in \tilde{T}} \{(r,(r,a,(p,s))),((r,a,(p,s)),p),((r,a,(p,s)),s)\} \\
&\qquad \subset (Q_{\mathbb{V}} \times \tilde{T}) \cup (\tilde{T} \times (Q \cup Q_{\mathbb{V}})), \\
\mathring{K} &:= \bigcup_{((q,r),a,(p,s)) \in \mathring{T}} \{(q,((q,r),a,(p,s))),(r,((q,r),a,(p,s))),(((q,r),a,(p,s)),p), \\
&\qquad (((q,r),a,(p,s)),s)\} \subset ((Q \cup Q_{\mathbb{V}}) \times \mathring{T}) \cup (\mathring{T} \times (Q \cup Q_{\mathbb{V}})), \\
\bar{K} &:= \bigcup_{((q,r),a,s) \in \bar{T}} \{(q,((q,r),a,s)),(r,((q,r),a,s)),(((q,r),a,s),s)\} \\
&\qquad \subset ((Q \cup Q_{\mathbb{V}}) \times \bar{T}) \cup (\bar{T} \times Q_{\mathbb{V}}), \text{ and} \\
\tilde{\bar{K}} &:= \bigcup_{(r,a,s) \in \tilde{\bar{T}}} \{(r,(r,a,s)),((r,a,s),s)\} \subset (Q_{\mathbb{V}} \times \tilde{\bar{T}}) \cup (\tilde{\bar{T}} \times Q_{\mathbb{V}}).
\end{aligned} \tag{162}$$

With these definitions of $N_{\mathbb{PV}}$, ι and χ the following can be shown by induction:

$$\begin{aligned}
&\text{For each } o = o_1 \ldots o_{|o|} \in (\mathbb{N}_0^{Q \cup Q_{\mathbb{V}}} \times T \times \mathbb{N}_0^{Q \cup Q_{\mathbb{V}}})^+ \\
&\text{with } o_i \in \mathbb{N}_0^{Q \cup Q_{\mathbb{V}}} \times T \times \mathbb{N}_0^{Q \cup Q_{\mathbb{V}}} \text{ for } 1 \le i \le |o| \text{ holds } o \in \mathcal{I}^{-1}(\iota((0,q_{\mathbb{V}_0}))), \\
&\text{iff there exists } x \in \alpha_{\mathbb{P}}^{-1}(\text{pre}(V)) \text{ with } |x| = |o| \text{ such that for } 1 \le i \le |o| \text{ holds:} \\
&o_i = (\iota(Z_{\mathbb{PV}}(x'_{i-1})), t_i, \iota(Z_{\mathbb{PV}}(x'_i))) \text{ with } x'_j \in \text{pre}(x) \text{ and } \left|x'_j\right| = j \text{ for } 0 \le j \le |o| \\
&\text{and} \\
&t_i \in \chi((y_i, \delta_{\mathbb{V}}(q_{\mathbb{V}0}, \alpha_{\mathbb{P}}(x'_{i-1})))) \text{ with } y_i \in \sigma_{\mathbb{P}}(x_i), \\
&\text{where } x = x_1 \ldots x_{|o|} \text{ and } x_i \in \sqcup\!\sqcup_{\mathbb{P}} \text{ for } 1 \le i \le |o|.
\end{aligned} \tag{163}$$

This proves (158). Because ι is injective, $Q_{\mathbb{PV}}$ is finite iff $\mathcal{E}(\iota((0,q_{\mathbb{V}_0})))$ is finite. The finiteness of $\mathcal{E}(M)$ is decidable for each each Petri net and each marking M of the net [12] [13]. Therefore, with (149) and (145), the following theorem holds.

Theorem 17. *If* $\mathbb{P}$ *and* $\mathbb{V}$ *are finite automata, then it is decidable if* $\text{alph}(\alpha_{\mathbb{P}}^{-1}(\text{pre}(V)))$ *is finite. In the positive case* $\text{alph}(\alpha_{\mathbb{P}}^{-1}(\text{pre}(V)))$ *is computable.*

The key to decide finiteness of $\mathcal{E}(M)$ is Dickson's lemma [8], [12]. Therefore Theorem 17 can also be proven by directly applying Dickson's lemma. We used the simulation by a Petri net, because we also need this simulation to tackle the more general problem to decide the finiteness of $\mathrm{alph}(Z_{\mathbb{P}}^{-1}(0)\cap\alpha_{\mathbb{P}}^{-1}(V))$.

For this we make the same assumptions as in the respective problem regarding $\mathrm{alph}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))$ and additionally postulate the existence of $F_{\mathbb{V}}\subset Q_{\mathbb{V}}$ with

$$V=\{w\in\mathrm{pre}(V)|\delta_{\mathbb{V}}(q_{\mathbb{V}_0},w)\in F_{\mathbb{V}}\}. \tag{164}$$

Let therefore,

$$\begin{aligned}\breve{A}_{\mathbb{P}\mathbb{V}}:=&\mathrm{pre}(Z_{\mathbb{P}}^{-1}(0)\cap\alpha_{\mathbb{P}}^{-1}(V))=\\ &\{u\in\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))|Z_{\mathbb{P}\mathbb{V}}(u(u^{-1}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))))\cap\{0\}\times F_{\mathbb{V}}\neq\emptyset\}.\end{aligned} \tag{165}$$

From (163) it follows:

$$\begin{aligned}&\text{For each } u\in\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \text{ holds}\\ &\iota(Z_{\mathbb{P}\mathbb{V}}(u(u^{-1}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V))))))=\mathcal{E}(\iota(Z_{\mathbb{P}\mathbb{V}}(u))).\end{aligned} \tag{166}$$

For each Petri net and each two markings M and M' it is decidable if $M'\in\mathcal{E}(M)$ [12], [13]. From this it follows on account of (166):

$$\text{For each } u\in\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) \text{ it is decidable, if } u\in\breve{A}_{\mathbb{P}\mathbb{V}}. \tag{167}$$

On account of (165):

$$\mathrm{alph}(Z_{\mathbb{P}}^{-1}(0)\cap\alpha_{\mathbb{P}}^{-1}(V))=\mathrm{alph}(\breve{A}_{\mathbb{P}\mathbb{V}}). \tag{168}$$

$$\text{Let now } \breve{Q}_{\mathbb{P}\mathbb{V}}:=Z_{\mathbb{P}\mathbb{V}}(\breve{A}_{\mathbb{P}\mathbb{V}})\subset Q_{\mathbb{P}\mathbb{V}}. \tag{169}$$

Analog to (146) and (147),

$$\mathrm{alph}(\breve{A}_{\mathbb{P}\mathbb{V}}) \text{ is finite if } \breve{Q}_{\mathbb{P}\mathbb{V}} \text{ is finite.} \tag{170}$$

$$\text{For each } n\in\mathbb{N}_0 \text{ let } \breve{Q}_{\mathbb{P}\mathbb{V}}^{(n)}:=\breve{Q}_{\mathbb{P}\mathbb{V}}\cap Q_{\mathbb{P}\mathbb{V}}^{(n)}. \tag{171}$$

Therewith,

$$\breve{Q}_{\mathbb{P}\mathbb{V}}^{(n)} \text{ are finite sets that are computable on account of (167).} \tag{172}$$

As in (144) - (148),

$$\begin{aligned}&\text{the stepwise computation of each } \breve{Q}_{\mathbb{P}\mathbb{V}}^{(i)} \text{ for } i\in\mathbb{N}_0 \text{ and the test } \breve{Q}_{\mathbb{P}\mathbb{V}}^{(i)}=\breve{Q}_{\mathbb{P}\mathbb{V}}^{(i+1)}\\ &\text{provides a semi-algorithm to decide the finiteness of } \mathrm{alph}(\breve{A}_{\mathbb{P}\mathbb{V}}).\end{aligned} \tag{173}$$

Let $k \in \mathbb{N}_0$ be the smallest $i \in \mathbb{N}_0$ such that $\breve{Q}^{(i)}_{\mathbb{P}\mathbb{V}} = \breve{Q}^{(i+1)}_{\mathbb{P}\mathbb{V}}$, then

$$\mathrm{alph}(\breve{A}_{\mathbb{P}\mathbb{V}}) = \mathrm{alph}(\breve{A}_{\mathbb{P}\mathbb{V}} \cap A^{(i+1)}_{\mathbb{P}}). \tag{174}$$

With regard to (170) it remains to prove the decidability of finiteness of $\breve{Q}_{\mathbb{P}\mathbb{V}}$. This can be done using the following result for Petri nets:

$$\begin{aligned}&\text{Let } M \text{ and } M' \text{ markings of a Petri net, then it is decidable if}\\ &\{\mathring{M} \in \mathcal{E}(M) | M' \in \mathcal{E}(\mathring{M})\} \text{ is finite [13].}\end{aligned} \tag{175}$$

On account of (166), (165), and (169):

$$\begin{aligned}&\breve{Q}_{\mathbb{P}\mathbb{V}} \text{ is finite iff for each } q \in F_{\mathbb{V}}\\ &\iota(Z_{\mathbb{P}\mathbb{V}}(\{u \in \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) | (0,q) \in Z_{\mathbb{P}\mathbb{V}}(u(u^{-1}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))))\}) \text{ is finite .}\end{aligned} \tag{176}$$

On account of (166) furthermore holds:

$$\begin{aligned}&\iota(Z_{\mathbb{P}\mathbb{V}}(\{u \in \alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)) | (0,q) \in Z_{\mathbb{P}\mathbb{V}}(u(u^{-1}(\alpha_{\mathbb{P}}^{-1}(\mathrm{pre}(V)))))\})\\ &= \{x \in \mathcal{E}(\iota((0, q_{\mathbb{V}_0}))) | \iota(0,q) \in \mathcal{E}(x)\}.\end{aligned} \tag{177}$$

Now (175) - (177) prove the following theorem:

Theorem 18. *If* $\mathbb{P}$ *and* $\mathbb{V}$ *are finite automata, then it is decidable if* $\mathrm{alph}(Z_{\mathbb{P}}^{-1}(0) \cap \alpha_{\mathbb{P}}^{-1}(V))$ *is finite. In the positive case* $\mathrm{alph}(Z_{\mathbb{P}}^{-1}(0) \cap \alpha_{\mathbb{P}}^{-1}(V))$ *is computable by* (174)*.*

Now, combining the technique of Section 7 with the simulation of S-automata by Petri nets will result in a proof of the decidability of $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ for regular P and V. The idea is, to consider the counterexamples for

$$\mathcal{R}'_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0)) \subset \alpha_{\mathbb{P}}^{-1}(V).$$

Preliminarily we notice that on account of (64d)

$$Z_{\mathbb{P}}(d) \leq Z_{\mathbb{P}}(c) \text{ for each } d \in \mathcal{R}'_{\mathbb{P}}(\{c\}). \tag{178}$$

By Corollary 9 $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ does not hold, iff there exists $c \in \alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0)$ and $d \in \mathcal{R}'_{\mathbb{P}}(\{c\})$ with $d \notin \alpha_{\mathbb{P}}^{-1}(V)$. With Theorem 16 this is equivalent to:

$$\begin{aligned}&\text{There exists } x \in W_{\sqcup\mathbb{P}} \text{ with}\\ &\mu_{\sqcup\mathbb{P}}(x) \in Z_{\mathbb{P}}^{-1}(0) \cap \alpha_{\mathbb{P}}^{-1}(V) \text{ and } \nu_{\sqcup\mathbb{P}}(x) \notin \alpha_{\mathbb{P}}^{-1}(V).\end{aligned} \tag{179}$$

As $\nu_{\sqcup\mathbb{P}}(x) \in \mathcal{R}'_{\mathbb{P}}(\{\mu_{\sqcup\mathbb{P}}(x)\})$ by (178) (179) is equivalent to

$$\begin{aligned}&\text{There exists } x \in W_{\sqcup\mathbb{P}} \text{ with}\\ &\mu_{\sqcup\mathbb{P}}(x) \in Z_{\mathbb{P}}^{-1}(0) \cap \alpha_{\mathbb{P}}^{-1}(V) \text{ and } \nu_{\sqcup\mathbb{P}}(x) \in Z_{\mathbb{P}}^{-1}(0) \setminus \alpha_{\mathbb{P}}^{-1}(V).\end{aligned} \tag{180}$$

The first step to decide $\mathrm{SP}(P\cup\{\varepsilon\},V)$ is to "isomorphically refine" the prefix closed language $W_{\sqcup_{\mathbb{P}}}$ by appropriately attaching states of $\mathbb{V}$ to the elements of $\sqcup_{\mathbb{P}}^{()}$. For that purpose, additionally to the assumptions about $\mathbb{P}$ and $\mathbb{V}$, we assume that $\mathbb{V}$ is complete. This means $\delta_{\mathbb{V}}: Q_{\mathbb{V}}\times\Sigma\to Q_{\mathbb{V}}$ is a total function, and it poses no restriction on V, as using an additional dummy state each deterministic automaton can be transformed into a complete deterministic automaton recognizing the same language.

According to $\sqcup_{\mathbb{P}}^{()}$, $\varphi_{\sqcup_{\mathbb{P}}}^{(1.2)}$ and $\varphi_{\sqcup_{\mathbb{P}}}^{(2.2)}$, as defined in (127), (102) and (124), let now

$$\begin{aligned}\Psi_{\mathbb{P}}^{\mathbb{V}} := \{((y_1,y_2),x,(y_1',y_2')) \in (Q_{\mathbb{V}}\times Q_{\mathbb{V}})\times\sqcup_{\mathbb{P}}^{()}\times(Q_{\mathbb{V}}\times Q_{\mathbb{V}})|\\ y_1' = \delta_{\mathbb{V}}(y_1,\varphi_{\sqcup_{\mathbb{P}}}^{(1.2)}(x_1)),\\ y_2' = \delta_{\mathbb{V}}(y_2,\varphi_{\sqcup_{\mathbb{P}}}^{(2.2)}(x_2)) \text{ if } x_2\in\sqcup_{\mathbb{P}}^{(2)'} \text{ and}\\ y_2' = y_2 \text{ if } x_2\in S_{\sqcup_{\mathbb{P}}}^{(2)}, \text{ with}\\ x=(x_1,x_2,x_3)\},\end{aligned} \tag{181}$$

let the mappings $Z_{\mathbb{P}}^{\mathbb{V}(1)}:\Psi_{\mathbb{P}}^{\mathbb{V}*}\to Q_{\mathbb{V}}$ and $Z_{\mathbb{P}}^{\mathbb{V}(2)}:\Psi_{\mathbb{P}}^{\mathbb{V}*}\to Q_{\mathbb{V}}$ be defined by

$$Z_{\mathbb{P}}^{\mathbb{V}(1)}(\varepsilon) := Z_{\mathbb{P}}^{\mathbb{V}(2)}(\varepsilon) := q_{\mathbb{V}_0},\ Z_{\mathbb{P}}^{\mathbb{V}(1)}(uv) := y_1' \text{ and } Z_{\mathbb{P}}^{\mathbb{V}(2)}(uv) := y_2' \tag{182}$$

for $u\in\Psi_{\mathbb{P}}^{\mathbb{V}*}$ and $v\in\Psi_{\mathbb{P}}^{\mathbb{V}}$ with $v=((y_1,y_2),x,(y_1',y_2'))$,

and let the homomorphism $\psi_{\mathbb{P}}^{\mathbb{V}}:\Psi_{\mathbb{P}}^{\mathbb{V}*}\to\sqcup_{\mathbb{P}}^{()*}$ be defined by

$$\psi_{\mathbb{P}}^{\mathbb{V}}((y_1,y_2),x,(y_1',y_2'))) := x \text{ for } ((y_1,y_2),x,(y_1',y_2'))\in\Psi_{\mathbb{P}}^{\mathbb{V}}. \tag{183}$$

Definition 27.
Let the prefix closed language $W_{\mathbb{P}}^{\mathbb{V}}\subset\Psi_{\mathbb{P}}^{\mathbb{V}*}$ be defined by

$$\begin{aligned}W_{\mathbb{P}}^{\mathbb{V}} := \{w\in(\psi_{\mathbb{P}}^{\mathbb{V}})^{-1}(W_{\sqcup_{\mathbb{P}}})|\ y_1 = Z_{\mathbb{P}}^{\mathbb{V}(1)}(u) \textit{ and } y_2 = Z_{\mathbb{P}}^{\mathbb{V}(2)}(u) \textit{ for each}\\ uv\in\mathrm{pre}(w) \textit{ with } u\in\Psi_{\mathbb{P}}^{\mathbb{V}*} \textit{ and } v=((y_1,y_2),x,(y_1',y_2'))\in\Psi_{\mathbb{P}}^{\mathbb{V}}\}.\end{aligned}$$

Now Definition 26 implies

$$\begin{aligned}Z_{\mathbb{P}}^{\mathbb{V}(1)}(u) = \delta_{\mathbb{V}}(q_{\mathbb{V}_0},\alpha_{\mathbb{P}}(\mu_{\sqcup_{\mathbb{P}}}(\psi_{\mathbb{P}}^{\mathbb{V}}(u)))) \text{ and}\\ Z_{\mathbb{P}}^{\mathbb{V}(2)}(u) = \delta_{\mathbb{V}}(q_{\mathbb{V}_0},\alpha_{\mathbb{P}}(\nu_{\sqcup_{\mathbb{P}}}(\psi_{\mathbb{P}}^{\mathbb{V}}(u)))) \text{ for each } u\in W_{\mathbb{P}}^{\mathbb{V}}.\end{aligned} \tag{184}$$

$$\text{As } \mathbb{V} \text{ is a complete deterministic automaton } (\psi_{\mathbb{P}}^{\mathbb{V}})_{|W_{\mathbb{P}}^{\mathbb{V}}} \text{ is a bijection.} \tag{185}$$

On account of (185), (180) is equivalent to:

There exists $u\in W_{\mathbb{P}}^{\mathbb{V}}$ with
$\mu_{\sqcup_{\mathbb{P}}}(\psi_{\mathbb{P}}^{\mathbb{V}}(u))\in Z_{\mathbb{P}}^{-1}(0)\cap\alpha_{\mathbb{P}}^{-1}(V)$ and $\nu_{\sqcup_{\mathbb{P}}}(\psi_{\mathbb{P}}^{\mathbb{V}}(u))\in Z_{\mathbb{P}}^{-1}(0)\setminus\alpha_{\mathbb{P}}^{-1}(V)$,

which by (184) can be equivalently restated in terms of reachable states:

$$\begin{aligned}&\text{There exists } u \in W_{\mathbb{P}}^{\mathbb{V}} \text{ with } Z_{\mathbb{P}}^{\mathbb{V}(1)}(u) \in F_{\mathbb{V}},\ Z_{\mathbb{P}}^{\mathbb{V}(2)}(u) \in Q_{\mathbb{V}} \setminus F_{\mathbb{V}} \text{ and}\\ &Z_{\mathbb{P}}(\mu_{\sqcup_{\mathbb{P}}}(\psi_{\mathbb{P}}^{\mathbb{V}}(u))) = 0 = Z_{\mathbb{P}}(\nu_{\sqcup_{\mathbb{P}}}(\psi_{\mathbb{P}}^{\mathbb{V}}(u))).\end{aligned} \tag{186}$$

Caused by this formulation, the second step to decide $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ is to construct a deterministic semiautomaton $\mathbb{W}_{\mathbb{P}}^{\mathbb{V}}$ recognizing $W_{\mathbb{P}}^{\mathbb{V}}$. Generally $\mathbb{W}_{\mathbb{P}}^{\mathbb{V}}$ will be infinite. It is an immediate consequence of Definition 27 that

$$\begin{aligned}W_{\mathbb{P}}^{\mathbb{V}} =&(\psi_{\mathbb{P}}^{\mathbb{V}})^{-1}(W_{\sqcup_{\mathbb{P}}}) \setminus (X_{\mathbb{P}}^{\mathbb{V}} \Psi_{\mathbb{P}}^{\mathbb{V}*} \cup \Psi_{\mathbb{P}}^{\mathbb{V}*} Y_{\mathbb{P}}^{\mathbb{V}} \Psi_{\mathbb{P}}^{\mathbb{V}*}) \text{ with}\\ X_{\mathbb{P}}^{\mathbb{V}} =&\{((y_1,y_2),x,(y_1',y_2')) \in \Psi_{\mathbb{P}}^{\mathbb{V}} | y_1 \neq q_{\mathbb{V}_0} \text{ or } y_2 \neq q_{\mathbb{V}_0}\} \text{ and}\\ Y_{\mathbb{P}}^{\mathbb{V}} =&\{((y_1,y_2),x,(y_1',y_2'))((\bar{y}_1,\bar{y}_2),\bar{x},(\bar{y}_1',\bar{y}_2')) \in \Psi_{\mathbb{P}}^{\mathbb{V}} \Psi_{\mathbb{P}}^{\mathbb{V}} |\\ &y_1' \neq \bar{y}_1 \text{ or } y_2' \neq \bar{y}_2\}.\end{aligned} \tag{187}$$

Let now $\mathbb{W}_{\sqcup_{\mathbb{P}}} = (\sqcup_{\mathbb{P}}^{()}, S_{\sqcup_{\mathbb{P}}}^{()}, \lambda_{\sqcup_{\mathbb{P}}}, s_0)$ be a deterministic semiautomaton recognizing $W_{\sqcup_{\mathbb{P}}}$, where $\lambda_{\sqcup_{\mathbb{P}}} : S_{\sqcup_{\mathbb{P}}}^{()} \times \sqcup_{\mathbb{P}}^{()} \to S_{\sqcup_{\mathbb{P}}}^{()}$ is a partial function and $s_0 \in S_{\sqcup_{\mathbb{P}}}^{()}$. Generally $\mathbb{W}_{\sqcup_{\mathbb{P}}}$ is infinite. (187) implies that the following deterministic semiautomaton $\mathbb{W}_{\mathbb{P}}^{\mathbb{V}}$ recognizes $W_{\mathbb{P}}^{\mathbb{V}}$:

$$\begin{aligned}&\mathbb{W}_{\mathbb{P}}^{\mathbb{V}} := (\Psi_{\mathbb{P}}^{\mathbb{V}}, S_{\mathbb{P}}^{\mathbb{V}}, \lambda_{\mathbb{P}}^{\mathbb{V}}, q_{\mathbb{P}_0}^{\mathbb{V}}) \text{ where}\\ &S_{\mathbb{P}}^{\mathbb{V}} := Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times S_{\sqcup_{\mathbb{P}}}^{()},\ q_{\mathbb{P}_0}^{\mathbb{V}} := (q_{\mathbb{V}_0}, q_{\mathbb{V}_0}, s_0), \text{ and}\\ &\lambda_{\mathbb{P}}^{\mathbb{V}} : S_{\mathbb{P}}^{\mathbb{V}} \times \Psi_{\mathbb{P}}^{\mathbb{V}} \to S_{\mathbb{P}}^{\mathbb{V}} \text{ is a partial function with}\\ &\lambda_{\mathbb{P}}^{\mathbb{V}}((y_1,y_2,s),a) := (y_1',y_2',s'), \text{ for } (y_1,y_2,s) \in Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times S_{\sqcup_{\mathbb{P}}}^{()},\\ &a = ((y_1,y_2),x,(y_1',y_2')) \in \Psi_{\mathbb{P}}^{\mathbb{V}} \text{ and}\\ &\lambda_{\sqcup_{\mathbb{P}}}(s,x) = s'.\end{aligned} \tag{188}$$

By (188) and Definition 27 holds

$$\lambda_{\mathbb{P}}^{\mathbb{V}}(q_{\mathbb{P}_0}^{\mathbb{V}}, u) = (Z_{\mathbb{P}}^{\mathbb{V}(1)}(u), Z_{\mathbb{P}}^{\mathbb{V}(2)}(u), \lambda_{\sqcup_{\mathbb{P}}}(s_0, \psi_{\mathbb{P}}^{\mathbb{V}}(u))) \text{ for each } u \in W_{\mathbb{P}}^{\mathbb{V}}. \tag{189}$$

To completely define $\mathbb{W}_{\mathbb{P}}^{\mathbb{V}}$, a complete definition of $\mathbb{W}_{\sqcup_{\mathbb{P}}}$ must be given. For that purpose we need the mapping $\check{\varphi}_{\sqcup_{\mathbb{P}}}^{(3.1)} : \sqcup_{\mathbb{P}}^{(3)} \to S_{\sqcup_{\mathbb{P}}}^{(3)} \dot\cup \{\check{0}\}$ defined by

$$\begin{aligned}&\check{\varphi}_{\sqcup_{\mathbb{P}}}^{(3.1)}((f,a,g)) := f \text{ for } (f,a,g) \in \sqcup_{\mathbb{P}}^{(3)'},\\ &\check{\varphi}_{\sqcup_{\mathbb{P}}}^{(3.1)}(f) := f \text{ for } f \in S_{\sqcup_{\mathbb{P}}}^{(3)} \text{ and}\\ &\check{\varphi}_{\sqcup_{\mathbb{P}}}^{(3.1)}(\check{0}) := \check{0}.\end{aligned} \tag{190}$$

As in Example 13, (127) and Definition 25 can be directly translated into a deterministic semiautomaton $\mathbb{W}_{\sqcup_{\mathbb{P}}}$. Let therefore

$$
\begin{aligned}
&S^{()}_{\sqcup_{\mathbb{P}}} := S^{(1)}_{\sqcup_{\mathbb{P}}} \times S^{(2)}_{\sqcup_{\mathbb{P}}} \times (S^{(3)}_{\sqcup_{\mathbb{P}}} \dot\cup \{\check{0}\}),\ s_0 := (0,0,0), \text{ and let}\\
&\lambda_{\sqcup_{\mathbb{P}}}((q_1,q_2,q_3),(x_1,x_2,x_3)) \text{ be defined for}\\
&(q_1,q_2,q_3) \in S^{()}_{\sqcup_{\mathbb{P}}} \text{ and } (x_1,x_2,x_3) \in \sqcup^{()}_{\mathbb{P}} \text{ with}\\
&(q_1,q_2,q_3) = (\varphi^{(1.1)}_{\sqcup_{\mathbb{P}}}(x_1), \varphi^{(2.1)}_{\sqcup_{\mathbb{P}}}(x_2), \check{\varphi}^{(3.1)}_{\sqcup_{\mathbb{P}}}(x_3)), \text{ where}\\
&\lambda_{\sqcup_{\mathbb{P}}}((q_1,q_2,q_3),(x_1,x_2,x_3)) := (\varphi^{(1.3)}_{\sqcup_{\mathbb{P}}}(x_1), \varphi^{(2.3)}_{\sqcup_{\mathbb{P}}}(x_2), \check{0}) \text{ for}\\
&x_3 \in (\sqcup^{(3)'}_{\mathbb{P}} \cap (\varphi^{(3.3)}_{\sqcup_{\mathbb{P}}})^{-1}(0)) \cup \{\check{0}\} \text{ and}\\
&\lambda_{\sqcup_{\mathbb{P}}}((q_1,q_2,q_3),(x_1,x_2,x_3)) := (\varphi^{(1.3)}_{\sqcup_{\mathbb{P}}}(x_1), \varphi^{(2.3)}_{\sqcup_{\mathbb{P}}}(x_2), \varphi^{(3.3)}_{\sqcup_{\mathbb{P}}}(x_3)) \text{ for}\\
&x_3 \in \sqcup^{(3)}_{\mathbb{P}} \setminus ((\sqcup^{(3)'}_{\mathbb{P}} \cap (\varphi^{(3.3)}_{\sqcup_{\mathbb{P}}})^{-1}(0)) \cup \{\check{0}\}).
\end{aligned}
\tag{191}
$$

Now by induction it is easy to show that $\mathbb{W}_{\sqcup_{\mathbb{P}}}$ recognizes $W_{\sqcup_{\mathbb{P}}}$. With (96), (117), Definition 25 and Definition 26, (191) implies

$$
\begin{aligned}
&Z_{\mathbb{P}}(\mu_{\sqcup_{\mathbb{P}}}(w)) = q_1 \text{ and } Z_{\mathbb{P}}(\nu_{\sqcup_{\mathbb{P}}}(w)) = q_2, \text{ with } \lambda_{\sqcup_{\mathbb{P}}}((0,0,0),w) = (q_1,q_2,q_3),\\
&\text{for each } w \in W_{\sqcup_{\mathbb{P}}}.
\end{aligned}
\tag{192}
$$

By (126) holds

$$
\begin{aligned}
&\lambda_{\sqcup_{\mathbb{P}}}((0,0,0),w) \in \{(0,0,0),(0,0,\check{0})\} \text{ for each } w \in W_{\sqcup_{\mathbb{P}}} \text{ with}\\
&\lambda_{\sqcup_{\mathbb{P}}}((0,0,0),w) \in \{0\} \times S^{(2)}_{\sqcup_{\mathbb{P}}} \times (S^{(3)}_{\sqcup_{\mathbb{P}}} \dot\cup \{\check{0}\}).
\end{aligned}
\tag{193}
$$

Now, on account of (180), (186), (189), (192) and (193)

$$
\begin{aligned}
&\mathrm{SP}(P \cup \{\varepsilon\}, V) \text{ iff there don't exist any } u \in W^{\mathbb{V}}_{\mathbb{P}} \text{ with}\\
&\lambda^{\mathbb{V}}_{\mathbb{P}}(q^{\mathbb{V}}_{\mathbb{P}_0}, u) \in F_{\mathbb{V}} \times (Q_{\mathbb{V}} \setminus F_{\mathbb{V}}) \times \{(0,0,0),(0,0,\check{0})\}.
\end{aligned}
\tag{194}
$$

A more detailed analysis shows that

$$
\begin{aligned}
&\lambda^{\mathbb{V}}_{\mathbb{P}}(q^{\mathbb{V}}_{\mathbb{P}_0}, u) \in F_{\mathbb{V}} \times (Q_{\mathbb{V}} \setminus F_{\mathbb{V}}) \times \{(0,0,0),(0,0,\check{0})\} \text{ iff}\\
&\lambda^{\mathbb{V}}_{\mathbb{P}}(q^{\mathbb{V}}_{\mathbb{P}_0}, u) \in F_{\mathbb{V}} \times (Q_{\mathbb{V}} \setminus F_{\mathbb{V}}) \times \{(0,0,\check{0})\}.
\end{aligned}
$$

The reachability question posed by (194) can be decided by simulating $\mathbb{W}^{\mathbb{V}}_{\mathbb{P}}$ by a Petri net. Preparative to that simulation, first we need an appropriate characterization of $\lambda^{\mathbb{V}}_{\mathbb{P}}$, similar to the characterization of $\sqcup_{\mathbb{P}}$ by $\sqcup^{\sigma}_{\mathbb{P}}$ together with $\mathbb{N}^{Q}_{0}$. So the third step to decide $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ is to present such a characterization.

By (181), (182), (188) and (191) $\lambda^{\mathbb{V}}_{\mathbb{P}}$ is uniquely determined by $\delta_{\mathbb{V}}$ and $\sqcup^{()}_{\mathbb{P}}$. Therefore we now look for an appropriate characterization of $\sqcup^{()}_{\mathbb{P}}$. For that purpose we assume

$$
\mathrm{alph}(P) = \Sigma, \tag{195}
$$

which don't cause any restriction for P. Now we assemble the different sets to define $\sqcup\!\sqcup_{\mathbb{P}}^{()}$. On account of (99), (106), (107), (62c) and $E_{\mathbb{P}} \subset A_{\mathbb{P}}$ holds

$$\sqcup\!\sqcup_{\mathbb{P}}^{(1)} = \sqcup\!\sqcup_{\mathbb{P}},\ S_{\sqcup\!\sqcup_{\mathbb{P}}}^{(1)} = Z_{\mathbb{P}}(A_{\mathbb{P}}) \text{ and } S_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3)} = Z_{\mathbb{P}}(E_{\mathbb{P}}). \tag{196}$$

$$\begin{aligned} &S_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3)} \text{ is finite and can be effectively determined, if} \\ &\mathbb{P} \text{ is finite, and it holds } 0 \in S_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3)}. \end{aligned} \tag{197}$$

On account of (108), (197), (111), (195) and (118) it holds

$$\begin{aligned} &S_{\sqcup\!\sqcup_{\mathbb{P}}}^{(2)} = Z_{\mathbb{P}}(A_{\mathbb{P}}),\ \sqcup\!\sqcup_{\mathbb{P}}^{(2)'} = \sqcup\!\sqcup_{\mathbb{P}} \cap (Z_{\mathbb{P}}(A_{\mathbb{P}}) \times \Sigma \times Z_{\mathbb{P}}(A_{\mathbb{P}})) \text{ and} \\ &\sqcup\!\sqcup_{\mathbb{P}}^{(3)'} = \sqcup\!\sqcup_{\check{\mathbb{P}}}^{E} \cap (Z_{\mathbb{P}}(E_{\mathbb{P}}) \times \check{\iota}^{-1}(\Sigma) \times Z_{\mathbb{P}}(E_{\mathbb{P}})) = \sqcup\!\sqcup_{\check{\mathbb{P}}}^{\sigma}. \end{aligned} \tag{198}$$

$$\text{So } \sqcup\!\sqcup_{\mathbb{P}}^{(3)'} = \sqcup\!\sqcup_{\check{\mathbb{P}}}^{\sigma} \text{ is finite and can be effectively determined, if } \mathbb{P} \text{ is finite.} \tag{199}$$

By (132) (127) can be rephrased. Let therefore the mappings $\varphi_{\Delta}^{(1.1)'}$, $\varphi_{\Delta}^{(1.2)'}$ and $\varphi_{\Delta}^{(1.3)'}$ be defined by

$$\begin{aligned} &\varphi_{\Delta}^{(1.1)'} : \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \to \mathbb{N}_0^Q \text{ with } \varphi_{\Delta}^{(1.1)'}((f,a,g)) := f, \\ &\varphi_{\Delta}^{(1.2)'} : \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \to \Sigma \text{ with } \varphi_{\Delta}^{(1.2)'}((f,a,g)) := a, \\ &\varphi_{\Delta}^{(1.3)'} : \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \to \mathbb{N}_0^Q \text{ with } \varphi_{\Delta}^{(1.3)'}((f,a,g)) := g \\ &\text{for each } (f,a,g) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q. \end{aligned} \tag{200}$$

Then (127) becomes

$$\begin{aligned} \Delta^{()} = \{\ &(x_1,x_2,x_3) \in (\mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q) \times \Delta^{(2)} \times \Delta^{(3)} \mid \\ &\varphi_{\Delta}^{(1.1)'}(x_1) = \varphi_{\Delta}^{(2.1)}(x_2) + \varphi_{\Delta}^{(3.1)}(x_3), \\ &\varphi_{\Delta}^{(1.3)'}(x_1) = \varphi_{\Delta}^{(2.3)}(x_2) + \varphi_{\Delta}^{(3.3)}(x_3) \text{ and} \\ &\text{either } x_2 \in \Delta^{(2)'},\ x_3 \in S_{\Delta}^{(3)} \cup \{\check{0}\} \text{ and } \varphi_{\Delta}^{(1.2)'}(x_1) = \varphi_{\Delta}^{(2.2)}(x_2) \\ &\text{or } x_2 \in S_{\Delta}^{(2)},\ x_3 \in \Delta^{(3)'} \text{ and } \varphi_{\Delta}^{(1.2)'}(x_1) = \check{\iota}(\varphi_{\Delta}^{(3.2)}(x_3))\ \}. \end{aligned} \tag{201}$$

Now (201) together with (112), (118), (196) and (198) implies

$$\begin{aligned}
\sqcup\!\sqcup_{\mathbb{P}}^{()} &= \sqcup\!\sqcup_{\mathbb{P}}^{(S)} \,\dot\cup\, \sqcup\!\sqcup_{\mathbb{P}}^{(E)} \quad \text{with} \\
\sqcup\!\sqcup_{\mathbb{P}}^{(S)} &:= \{\ (x_1,x_2,x_3) \in (\mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q) \times \sqcup\!\sqcup_{\mathbb{P}}^{(2)'} \times (Z_{\mathbb{P}}(E_{\mathbb{P}}) \dot\cup \{\check{0}\}) \mid \\
&\qquad \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(1.1)'}(x_1) = \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(2.1)}(x_2) + \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3.1)}(x_3), \\
&\qquad \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(1.3)'}(x_1) = \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(2.3)}(x_2) + \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3.3)}(x_3) \text{ and} \\
&\qquad \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(1.2)'}(x_1) = \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(2.2)}(x_2)\ \} \text{ and} \\
\sqcup\!\sqcup_{\mathbb{P}}^{(E)} &:= \{\ (x_1,x_2,x_3) \in (\mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q) \times Z_{\mathbb{P}}(A_{\mathbb{P}}) \times \sqcup\!\sqcup_{\mathbb{P}}^{\sigma} \mid \\
&\qquad \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(1.1)'}(x_1) = \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(2.1)}(x_2) + \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3.1)}(x_3), \\
&\qquad \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(1.3)'}(x_1) = \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(2.3)}(x_2) + \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3.3)}(x_3) \text{ and} \\
&\qquad \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(1.2)'}(x_1) = \check{\imath}(\varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3.2)}(x_3))\ \}.
\end{aligned} \tag{202}$$

Because of (198) and (152) holds

$$\begin{aligned}
&(f,a,g) \in \sqcup\!\sqcup_{\mathbb{P}}^{(2)'} \text{ iff there exists } h \in \mathbb{N}_0^Q \text{ and } (f',a,g') \in \sqcup\!\sqcup_{\mathbb{P}}^{\sigma} \text{ with} \\
&f = f' + h \in Z_{\mathbb{P}}(A_{\mathbb{P}}) \text{ and } g = g' + h.
\end{aligned} \tag{203}$$

This implies

$$\begin{aligned}
\sqcup\!\sqcup_{\mathbb{P}}^{(S)} = \{\ &(x_1,x_2,x_3) \in (\mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q) \times \sqcup\!\sqcup_{\mathbb{P}}^{(2)'} \times (Z_{\mathbb{P}}(E_{\mathbb{P}}) \dot\cup \{\check{0}\}) \mid \\
&\text{there exist } (f',a,g') \in \sqcup\!\sqcup_{\mathbb{P}}^{\sigma} \text{ and } h \in \mathbb{N}_0^Q, \text{ such that} \\
&f' + h \in Z_{\mathbb{P}}(A_{\mathbb{P}}),\ x_2 = (f'+h, a, g'+h) \text{ and} \\
&x_1 = (f' + h + \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3.1)}(x_3), a, g' + h + \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3.3)}(x_3))\ \}.
\end{aligned} \tag{204}$$

Similar to (204) $\sqcup\!\sqcup_{\mathbb{P}}^{(E)}$ can be represented by

$$\begin{aligned}
\sqcup\!\sqcup_{\mathbb{P}}^{(E)} = \{\ &(x_1,x_2,x_3) \in (\mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q) \times Z_{\mathbb{P}}(A_{\mathbb{P}}) \times \sqcup\!\sqcup_{\check{\mathbb{P}}}^{\sigma} \mid \\
&\text{there exist } (f',\check{a},g') \in \sqcup\!\sqcup_{\check{\mathbb{P}}}^{\sigma} \text{ and } h \in Z_{\mathbb{P}}(A_{\mathbb{P}}), \text{ such that} \\
&x_3 = (f',\check{a},g'),\ x_2 = h \text{ and } x_1 = (f'+h, \check{\imath}(\check{a}), g'+h)\ \}.
\end{aligned} \tag{205}$$

On account of (153) the representation (204) is ambiguous. Contrary to (204), the representation (205) is unique. To capture the ambiguity of (204) let the mapping

$$\begin{aligned}
&\sigma_{\mathbb{P}}^{(S)} : \sqcup\!\sqcup_{\mathbb{P}}^{(S)} \to 2^{\sqcup\!\sqcup_{\mathbb{P}}^{\sigma}} \setminus \{\emptyset\} \text{ be defined by} \\
&\sigma_{\mathbb{P}}^{(S)}((x_1,x_2,x_3)) := \{(f',a,g') \in \sqcup\!\sqcup_{\mathbb{P}}^{\sigma} \mid \text{there exists } h \in \mathbb{N}_0^Q \text{ such that} \\
&\qquad f' + h \in Z_{\mathbb{P}}(A_{\mathbb{P}}),\ x_2 = (f'+h, a, g'+h) \text{ and} \\
&\qquad x_1 = (f' + h + \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3.1)}(x_3), a, g' + h + \varphi_{\sqcup\!\sqcup_{\mathbb{P}}}^{(3.3)}(x_3))\ \} \\
&\text{for each } (x_1,x_2,x_3) \in \sqcup\!\sqcup_{\mathbb{P}}^{(S)}.
\end{aligned} \tag{206}$$

As $\sqcup_{\mathbb{P}}^{()} = \sqcup_{\mathbb{P}}^{(S)} \mathbin{\dot\cup} \sqcup_{\mathbb{P}}^{(E)}$, for technical reasons $\sigma_{\mathbb{P}}^{(S)}$ can be extended to

$$\begin{aligned}
&\sigma_{\mathbb{P}}^{()} : \sqcup_{\mathbb{P}}^{()} \to (2^{\sqcup_{\mathbb{P}}^{\sigma}} \setminus \{\emptyset\}) \mathbin{\dot\cup} (2^{\sqcup_{\mathbb{P}}^{\sigma}} \setminus \{\emptyset\}) \text{ by} \\
&\sigma_{\mathbb{P}}^{()}((x_1,x_2,x_3)) := \sigma_{\mathbb{P}}^{(S)}((x_1,x_2,x_3)) \qquad \text{for } (x_1,x_2,x_3) \in \sqcup_{\mathbb{P}}^{(S)} \text{ and} \\
&\sigma_{\mathbb{P}}^{()}((x_1,x_2,x_3)) := \{(f',\check{a},g') \in \sqcup_{\mathbb{P}}^{\sigma} \mid \text{there exists } h \in Z_{\mathbb{P}}(A_{\mathbb{P}}) \text{ such that} \\
&\qquad x_3 = (f',\check{a},g'),\ x_2 = h \text{ and } x_1 = (f'+h, \check{\imath}(\check{a}), g'+h)\ \} \\
&\qquad\qquad \text{for } (x_1,x_2,x_3) \in \sqcup_{\mathbb{P}}^{(E)}, \\
&\text{which implies } \#(\sigma_{\mathbb{P}}^{()}((x_1,x_2,x_3))) = 1 \text{ for } (x_1,x_2,x_3) \in \sqcup_{\mathbb{P}}^{(E)}.
\end{aligned} \tag{207}$$

Now (204), (205) and (207) present an appropriate characterization of $\sqcup_{\mathbb{P}}^{()}$ to simulate $\mathbb{W}_{\mathbb{P}}^{\mathbb{V}}$ by a Petri net $N_{\mathbb{P}}^{\mathbb{V}}$, which is the final step to decide $\mathrm{SP}(P \cup \{\varepsilon\}, V)$. For that purpose we additionally assume finiteness of $\mathbb{P}$ and $\mathbb{V}$. To define the set of places of $N_{\mathbb{P}}^{\mathbb{V}}$, let

$$\begin{aligned}
&Q^{(i)} \text{ and } Q_{\mathbb{V}}^{(i)} \text{ for each } i \in \{1,2\} \text{ be copies of } Q \text{ and } Q_{\mathbb{V}} \text{ with} \\
&Q^{(1)} \cap Q^{(2)} = \emptyset = Q_{\mathbb{V}}^{(1)} \cap Q_{\mathbb{V}}^{(2)} \text{ and } Q^{(i)} \cap Q_{\mathbb{V}}^{(j)} = \emptyset \text{ for each } i,j \in \{1,2\}, \text{ and let} \\
&\tau^{(i)} : Q^{(i)} \mathbin{\dot\cup} Q_{\mathbb{V}}^{(i)} \to Q \mathbin{\dot\cup} Q_{\mathbb{V}} \text{ for each } i \in \{1,2\} \text{ be the corresponding bijections} \\
&\text{with } \tau^{(i)}(Q^{(i)}) = Q \text{ and } \tau^{(i)}(Q_{\mathbb{V}}^{(i)}) = Q_{\mathbb{V}} \text{ for each } i \in \{1,2\}.
\end{aligned} \tag{208}$$

Corresponding to the state set $S_{\mathbb{P}}^{\mathbb{V}}$ of the semiautomaton $W_{\mathbb{P}}^{\mathbb{V}}$, which by (188), (191), (196) and (198) is represented by

$$S_{\mathbb{P}}^{\mathbb{V}} = Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times (Z_{\mathbb{P}}(A_{\mathbb{P}}) \times Z_{\mathbb{P}}(A_{\mathbb{P}}) \times (Z_{\mathbb{P}}(E_{\mathbb{P}}) \mathbin{\dot\cup} \{\check{0}\})),$$

the set $R_{\mathbb{P}}^{\mathbb{V}}$ of places of $N_{\mathbb{P}}^{\mathbb{V}}$ is defined by

$$R_{\mathbb{P}}^{\mathbb{V}} := Q_{\mathbb{V}}^{(1)} \mathbin{\dot\cup} Q_{\mathbb{V}}^{(2)} \mathbin{\dot\cup} (Q^{(1)} \mathbin{\dot\cup} Q^{(2)} \mathbin{\dot\cup} (Z_{\mathbb{P}}(E_{\mathbb{P}}) \mathbin{\dot\cup} \{\check{0}\})). \tag{209}$$

By this definition there exists an injective mapping from $S_{\mathbb{P}}^{\mathbb{V}}$ into the set of markings of $N_{\mathbb{P}}^{\mathbb{V}}$. Let therefore the injection

$$\begin{aligned}
&\iota_{\mathbb{P}}^{\mathbb{V}}: S_{\mathbb{P}}^{\mathbb{V}} \to \mathbb{N}_0^{Q_{\mathbb{V}}^{(1)} \cup Q_{\mathbb{V}}^{(2)} \cup (Q^{(1)} \cup Q^{(2)} \cup (Z_{\mathbb{P}}(E_{\mathbb{P}}) \cup \{\check{0}\}))} \text{ be defined by}\\
&\iota_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(s_1,s_2,s_3)))(x) := 0 \text{ for } x \in Q_{\mathbb{V}}^{(1)} \setminus \{(\tau^{(1)})^{-1}(q_1)\},\\
&\iota_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(s_1,s_2,s_3)))(x) := 1 \text{ for } x \in Q_{\mathbb{V}}^{(1)} \cap \{(\tau^{(1)})^{-1}(q_1)\},\\
&\iota_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(s_1,s_2,s_3)))(x) := 0 \text{ for } x \in Q_{\mathbb{V}}^{(2)} \setminus \{(\tau^{(2)})^{-1}(q_2)\},\\
&\iota_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(s_1,s_2,s_3)))(x) := 1 \text{ for } x \in Q_{\mathbb{V}}^{(2)} \cap \{(\tau^{(2)})^{-1}(q_2)\},\\
&\iota_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(s_1,s_2,s_3)))(x) := s_1(\tau^{(1)}(x)) \text{ for } x \in Q^{(1)},\\
&\iota_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(s_1,s_2,s_3)))(x) := s_2(\tau^{(2)}(x)) \text{ for } x \in Q^{(2)},\\
&\iota_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(s_1,s_2,s_3)))(x) := 0 \text{ for } x \in (Z_{\mathbb{P}}(E_{\mathbb{P}}) \cup \{\check{0}\}) \setminus \{s_3\}, \text{ and}\\
&\iota_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(s_1,s_2,s_3)))(x) := 1 \text{ for } x \in (Z_{\mathbb{P}}(E_{\mathbb{P}}) \cup \{\check{0}\}) \cap \{s_3\} \text{ for each}\\
&(q_1,q_2,(s_1,s_2,s_3)) \in S_{\mathbb{P}}^{\mathbb{V}} \subset Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times (\mathbb{N}_0^Q \times \mathbb{N}_0^Q \times (Z_{\mathbb{P}}(E_{\mathbb{P}}) \cup \{\check{0}\})). \qquad (210)
\end{aligned}$$

The set $T_{\mathbb{P}}^{\mathbb{V}}$ of transitions of $N_{\mathbb{P}}^{\mathbb{V}}$ will be defined such that there exists a bijective mapping $\chi_{\mathbb{P}}^{\mathbb{V}}: (Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times \sqcup\!\sqcup_{\mathbb{P}}^{\sigma}) \cup (Q_{\mathbb{V}} \times \sqcup\!\sqcup_{\check{\mathbb{P}}}^{\sigma}) \to T_{\mathbb{P}}^{\mathbb{V}}$. For this purpose let

$$T_{\mathbb{P}}^{\mathbb{V}} := \tilde{T}_{\mathbb{P}}^{\mathbb{V}(S)} \cup \mathring{T}_{\mathbb{P}}^{\mathbb{V}(S)} \cup \bar{T}_{\mathbb{P}}^{\mathbb{V}(S)} \cup \tilde{\bar{T}}_{\mathbb{P}}^{\mathbb{V}(S)} \cup \tilde{T}_{\mathbb{P}}^{\mathbb{V}(E)} \cup \mathring{T}_{\mathbb{P}}^{\mathbb{V}(E)} \cup \bar{T}_{\mathbb{P}}^{\mathbb{V}(E)} \cup \tilde{\bar{T}}_{\mathbb{P}}^{\mathbb{V}(E)}, \text{ where}$$

$$\begin{aligned}
\tilde{T}_{\mathbb{P}}^{\mathbb{V}(S)} :=& \{((q_1,q_2),(a,p),(p_1,p_2)) \in (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) \times (\Sigma \times Q) \times (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) |\\
& (0,a,1_p) \in \tilde{\sqcup\!\sqcup}_{\mathbb{P}}^{\sigma},\ \delta_{\mathbb{V}}(q_1,a) = p_1 \text{ and } \delta_{\mathbb{V}}(q_2,a) = p_2\},\\
\mathring{T}_{\mathbb{P}}^{\mathbb{V}(S)} :=& \{((q_1,q_2),(q,a,p),(p_1,p_2)) \in (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) \times (Q \times \Sigma \times Q) \times (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) |\\
& (1_q,a,1_p) \in \mathring{\sqcup\!\sqcup}_{\mathbb{P}}^{\sigma},\ \delta_{\mathbb{V}}(q_1,a) = p_1 \text{ and } \delta_{\mathbb{V}}(q_2,a) = p_2\},\\
\bar{T}_{\mathbb{P}}^{\mathbb{V}(S)} :=& \{((q_1,q_2),(q,a),(p_1,p_2)) \in (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) \times (Q \times \Sigma) \times (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) |\\
& (1_q,a,0) \in \bar{\sqcup\!\sqcup}_{\mathbb{P}}^{\sigma},\ \delta_{\mathbb{V}}(q_1,a) = p_1 \text{ and } \delta_{\mathbb{V}}(q_2,a) = p_2\},\\
\tilde{\bar{T}}_{\mathbb{P}}^{\mathbb{V}(S)} :=& \{((q_1,q_2),a,(p_1,p_2)) \in (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) \times \Sigma \times (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) |\\
& (0,a,0) \in \tilde{\bar{\sqcup\!\sqcup}}_{\mathbb{P}}^{\sigma},\ \delta_{\mathbb{V}}(q_1,a) = p_1 \text{ and } \delta_{\mathbb{V}}(q_2,a) = p_2\},\\
\tilde{T}_{\mathbb{P}}^{\mathbb{V}(E)} :=& \{(q_1,(\check{a},p),p_1) \in Q_{\mathbb{V}} \times (\check{\Sigma} \times Q) \times Q_{\mathbb{V}} |\\
& (0,\check{a},1_p) \in \tilde{\sqcup\!\sqcup}_{\check{\mathbb{P}}}^{\sigma} \text{ and } \delta_{\mathbb{V}}(q_1,a) = p_1\},\\
\mathring{T}_{\mathbb{P}}^{\mathbb{V}(E)} :=& \{(q_1,(q,\check{a},p),p_1) \in Q_{\mathbb{V}} \times (Q \times \check{\Sigma} \times Q) \times Q_{\mathbb{V}} |\\
& (1_q,\check{a},1_p) \in \mathring{\sqcup\!\sqcup}_{\check{\mathbb{P}}}^{\sigma} \text{ and } \delta_{\mathbb{V}}(q_1,a) = p_1\},\\
\bar{T}_{\mathbb{P}}^{\mathbb{V}(E)} :=& \{(q_1,(q,\check{a}),p_1) \in Q_{\mathbb{V}} \times (Q \times \check{\Sigma}) \times Q_{\mathbb{V}} |\\
& (1_q,\check{a},0) \in \bar{\sqcup\!\sqcup}_{\check{\mathbb{P}}}^{\sigma} \text{ and } \delta_{\mathbb{V}}(q_1,a) = p_1\} \text{ and}\\
\tilde{\bar{T}}_{\mathbb{P}}^{\mathbb{V}(E)} :=& \{(q_1,\check{a},p_1) \in Q_{\mathbb{V}} \times \check{\Sigma} \times Q_{\mathbb{V}} | (0,\check{a},0) \in \tilde{\bar{\sqcup\!\sqcup}}_{\check{\mathbb{P}}}^{\sigma} \text{ and } \delta_{\mathbb{V}}(q_1,a) = p_1\}. \qquad (211)
\end{aligned}$$

Let now the bijective mapping $\chi_{\mathbb{P}}^{\mathbb{V}} : (Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times \sqcup_{\mathbb{P}}^{\sigma}) \dot{\cup} (Q_{\mathbb{V}} \times \sqcup_{\check{\mathbb{P}}}^{\sigma}) \to T_{\mathbb{P}}^{\mathbb{V}}$ be defined by

$$
\begin{aligned}
\chi_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(0,a,1_p))) :=& ((q_1,q_2),(a,p),(\delta_{\mathbb{V}}(q_1,a),\delta_{\mathbb{V}}(q_2,a))) \\
& \text{for } (q_1,q_2,(0,a,1_p)) \in Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times \tilde{\sqcup}_{\mathbb{P}}^{\sigma}, \\
\chi_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(1_q,a,1_p))) :=& ((q_1,q_2),(q,a,p),(\delta_{\mathbb{V}}(q_1,a),\delta_{\mathbb{V}}(q_2,a))) \\
& \text{for } (q_1,q_2,(1_q,a,1_p)) \in Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times \mathring{\sqcup}_{\mathbb{P}}^{\sigma}, \\
\chi_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(1_q,a,0))) :=& ((q_1,q_2),(q,a),(\delta_{\mathbb{V}}(q_1,a),\delta_{\mathbb{V}}(q_2,a))) \\
& \text{for } (q_1,q_2,(1_q,a,0)) \in Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times \bar{\sqcup}_{\mathbb{P}}^{\sigma}, \\
\chi_{\mathbb{P}}^{\mathbb{V}}((q_1,q_2,(0,a,0))) :=& ((q_1,q_2),a,(\delta_{\mathbb{V}}(q_1,a),\delta_{\mathbb{V}}(q_2,a))) \\
& \text{for } (q_1,q_2,(0,a,0)) \in Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times \tilde{\tilde{\sqcup}}_{\mathbb{P}}^{\sigma}, \\
\chi_{\mathbb{P}}^{\mathbb{V}}((q_1,(0,\check{a},1_p))) :=& (q_1,(\check{a},p),\delta_{\mathbb{V}}(q_1,a)) \\
& \text{for } (q_1,(0,\check{a},1_p)) \in Q_{\mathbb{V}} \times \tilde{\sqcup}_{\check{\mathbb{P}}}^{\sigma}, \\
\chi_{\mathbb{P}}^{\mathbb{V}}((q_1,(1_q,\check{a},1_p))) :=& (q_1,(q,\check{a},p),\delta_{\mathbb{V}}(q_1,a)) \\
& \text{for } (q_1,(1_q,\check{a},1_p)) \in Q_{\mathbb{V}} \times \mathring{\sqcup}_{\check{\mathbb{P}}}^{\sigma}, \\
\chi_{\mathbb{P}}^{\mathbb{V}}((q_1,(1_q,\check{a},0))) :=& (q_1,(q,\check{a}),\delta_{\mathbb{V}}(q_1,a)) \\
& \text{for } (q_1,(1_q,\check{a},0)) \in Q_{\mathbb{V}} \times \bar{\sqcup}_{\check{\mathbb{P}}}^{\sigma} \text{ and} \\
\chi_{\mathbb{P}}^{\mathbb{V}}((q_1,(0,\check{a},0))) :=& (q_1,\check{a},\delta_{\mathbb{V}}(q_1,a)) \\
& \text{for } (q_1,(0,\check{a},0)) \in Q_{\mathbb{V}} \times \tilde{\tilde{\sqcup}}_{\check{\mathbb{P}}}^{\sigma} .
\end{aligned} \tag{212}
$$

The set $K_{\mathbb{P}}^{\mathbb{V}}$ of edges of $N_{\mathbb{P}}^{\mathbb{V}}$ let be defined by

$$
K_{\mathbb{P}}^{\mathbb{V}} := \tilde{K}_{\mathbb{P}}^{\mathbb{V}(S)} \dot{\cup} \mathring{K}_{\mathbb{P}}^{\mathbb{V}(S)} \dot{\cup} \bar{K}_{\mathbb{P}}^{\mathbb{V}(S)} \dot{\cup} \tilde{\tilde{K}}_{\mathbb{P}}^{\mathbb{V}(S)} \dot{\cup} \tilde{K}_{\mathbb{P}}^{\mathbb{V}(E)} \dot{\cup} \mathring{K}_{\mathbb{P}}^{\mathbb{V}(E)} \dot{\cup} \bar{K}_{\mathbb{P}}^{\mathbb{V}(E)} \dot{\cup} \tilde{\tilde{K}}_{\mathbb{P}}^{\mathbb{V}(E)},
$$

where

$$
\begin{aligned}
\tilde{K}_{\mathbb{P}}^{\mathbb{V}(S)} := & \bigcup_{((q_1,q_2),(a,p),(p_1,p_2)) \in \tilde{T}_{\mathbb{P}}^{\mathbb{V}(S)}} \\
& \{((\tau^{(1)})^{-1}(q_1),((q_1,q_2),(a,p),(p_1,p_2))), \\
& ((\tau^{(2)})^{-1}(q_2),((q_1,q_2),(a,p),(p_1,p_2))), \\
& (((q_1,q_2),(a,p),(p_1,p_2)),(\tau^{(1)})^{-1}(p_1)), \\
& (((q_1,q_2),(a,p),(p_1,p_2)),(\tau^{(2)})^{-1}(p_2)), \\
& (((q_1,q_2),(a,p),(p_1,p_2)),(\tau^{(1)})^{-1}(p)), \\
& (((q_1,q_2),(a,p),(p_1,p_2)),(\tau^{(2)})^{-1}(p))\} \subset \\
& ((Q_{\mathbb{V}}^{(1)} \dot{\cup} Q_{\mathbb{V}}^{(2)}) \times \tilde{T}_{\mathbb{P}}^{\mathbb{V}(S)}) \dot{\cup} (\tilde{T}_{\mathbb{P}}^{\mathbb{V}(S)} \times (Q_{\mathbb{V}}^{(1)} \dot{\cup} Q_{\mathbb{V}}^{(2)} \dot{\cup} Q^{(1)} \dot{\cup} Q^{(2)})),
\end{aligned} \tag{213}
$$

$$
\begin{aligned}
\mathring{K}_{\mathbb{P}}^{\mathbb{V}(S)} := & \bigcup_{((q_1,q_2),(q,a,p),(p_1,p_2))\in \mathring{T}_{\mathbb{P}}^{\mathbb{V}(S)}} \\
& \{((\tau^{(1)})^{-1}(q_1),((q_1,q_2),(q,a,p),(p_1,p_2))), \\
& ((\tau^{(2)})^{-1}(q_2),((q_1,q_2),(q,a,p),(p_1,p_2))), \\
& ((\tau^{(1)})^{-1}(q),((q_1,q_2),(q,a,p),(p_1,p_2))), \\
& ((\tau^{(2)})^{-1}(q),((q_1,q_2),(q,a,p),(p_1,p_2))), \\
& (((q_1,q_2),(q,a,p),(p_1,p_2)),(\tau^{(1)})^{-1}(p_1)), \\
& (((q_1,q_2),(q,a,p),(p_1,p_2)),(\tau^{(2)})^{-1}(p_2)), \\
& (((q_1,q_2),(q,a,p),(p_1,p_2)),(\tau^{(1)})^{-1}(p)), \\
& (((q_1,q_2),(q,a,p),(p_1,p_2)),(\tau^{(2)})^{-1}(p))\} \subset \\
& ((Q_{\mathbb{V}}^{(1)} \dot\cup Q_{\mathbb{V}}^{(2)} \dot\cup Q^{(1)} \dot\cup Q^{(2)}) \times \mathring{T}_{\mathbb{P}}^{\mathbb{V}(S)}) \dot\cup (\mathring{T}_{\mathbb{P}}^{\mathbb{V}(S)} \times (Q_{\mathbb{V}}^{(1)} \dot\cup Q_{\mathbb{V}}^{(2)} \dot\cup Q^{(1)} \dot\cup Q^{(2)})),
\end{aligned} \tag{214}
$$

$$
\begin{aligned}
\bar{K}_{\mathbb{P}}^{\mathbb{V}(S)} := & \bigcup_{((q_1,q_2),(q,a),(p_1,p_2))\in \bar{T}_{\mathbb{P}}^{\mathbb{V}(S)}} \\
& \{((\tau^{(1)})^{-1}(q_1),((q_1,q_2),(q,a),(p_1,p_2))), \\
& ((\tau^{(2)})^{-1}(q_2),((q_1,q_2),(q,a),(p_1,p_2))), \\
& ((\tau^{(1)})^{-1}(q),((q_1,q_2),(q,a),(p_1,p_2))), \\
& ((\tau^{(2)})^{-1}(q),((q_1,q_2),(q,a),(p_1,p_2))), \\
& (((q_1,q_2),(q,a,p),(p_1,p_2)),(\tau^{(1)})^{-1}(p_1)), \\
& (((q_1,q_2),(q,a,p),(p_1,p_2)),(\tau^{(2)})^{-1}(p_2))\} \subset \\
& ((Q_{\mathbb{V}}^{(1)} \dot\cup Q_{\mathbb{V}}^{(2)} \dot\cup Q^{(1)} \dot\cup Q^{(2)}) \times \bar{T}_{\mathbb{P}}^{\mathbb{V}(S)}) \dot\cup (\bar{T}_{\mathbb{P}}^{\mathbb{V}(S)} \times (Q_{\mathbb{V}}^{(1)} \dot\cup Q_{\mathbb{V}}^{(2)})),
\end{aligned} \tag{215}
$$

$$
\begin{aligned}
\tilde{\bar{K}}_{\mathbb{P}}^{\mathbb{V}(S)} := & \bigcup_{((q_1,q_2),a,(p_1,p_2))\in \tilde{\bar{T}}_{\mathbb{P}}^{\mathbb{V}(S)}} \\
& \{((\tau^{(1)})^{-1}(q_1),((q_1,q_2),a,(p_1,p_2))), \\
& ((\tau^{(2)})^{-1}(q_2),((q_1,q_2),a,(p_1,p_2))), \\
& (((q_1,q_2),a,(p_1,p_2)),(\tau^{(1)})^{-1}(p_1)), \\
& (((q_1,q_2),a,(p_1,p_2)),(\tau^{(2)})^{-1}(p_2))\} \subset \\
& ((Q_{\mathbb{V}}^{(1)} \dot\cup Q_{\mathbb{V}}^{(2)}) \times \tilde{\bar{T}}_{\mathbb{P}}^{\mathbb{V}(S)}) \dot\cup (\tilde{\bar{T}}_{\mathbb{P}}^{\mathbb{V}(S)} \times (Q_{\mathbb{V}}^{(1)} \dot\cup Q_{\mathbb{V}}^{(2)})),
\end{aligned} \tag{216}
$$

$$\tilde{K}_{\mathbb{P}}^{\mathbb{V}(E)} := \bigcup_{(q_1,(\check{a},p),p_1)\in\tilde{T}_{\mathbb{P}}^{\mathbb{V}(E)}} \begin{aligned}[t] &\{((\tau^{(1)})^{-1}(q_1),(q_1,(\check{a},p),p_1)),\\ &(0,(q_1,(\check{a},p),p_1)),\\ &((q_1,(\check{a},p),p_1),(\tau^{(1)})^{-1}(p_1)),\\ &((q_1,(\check{a},p),p_1),1_p),\\ &((q_1,(\check{a},p),p_1),(\tau^{(1)})^{-1}(p))\}\subset\end{aligned}$$
$$((Q_{\mathbb{V}}^{(1)}\cup(Z_{\mathbb{P}}(E_{\mathbb{P}})\cup\{\check{0}\}))\times\tilde{T}_{\mathbb{P}}^{\mathbb{V}(E)})\cup$$
$$(\tilde{T}_{\mathbb{P}}^{\mathbb{V}(E)}\times(Q_{\mathbb{V}}^{(1)}\cup(Z_{\mathbb{P}}(E_{\mathbb{P}})\cup\{\check{0}\})\cup Q^{(1)})), \tag{217}$$

$$\mathring{K}_{\mathbb{P}}^{\mathbb{V}(E)} := \bigcup_{(q_1,(q,\check{a},p),p_1)\in\mathring{T}_{\mathbb{P}}^{\mathbb{V}(E)}} \begin{aligned}[t] &\{((\tau^{(1)})^{-1}(q_1),(q_1,(q,\check{a},p),p_1)),\\ &(1_q,(q_1,(q,\check{a},p),p_1)),\\ &\{((\tau^{(1)})^{-1}(q),(q_1,(q,\check{a},p),p_1)),\\ &((q_1,(q,\check{a},p),p_1),(\tau^{(1)})^{-1}(p_1)),\\ &((q_1,(q,\check{a},p),p_1),1_p),\\ &((q_1,(q,\check{a},p),p_1),(\tau^{(1)})^{-1}(p))\}\subset\end{aligned}$$
$$((Q_{\mathbb{V}}^{(1)}\cup(Z_{\mathbb{P}}(E_{\mathbb{P}})\cup\{\check{0}\})\cup Q^{(1)})\times\mathring{T}_{\mathbb{P}}^{\mathbb{V}(E)})\cup$$
$$(\mathring{T}_{\mathbb{P}}^{\mathbb{V}(E)}\times(Q_{\mathbb{V}}^{(1)}\cup(Z_{\mathbb{P}}(E_{\mathbb{P}})\cup\{\check{0}\})\cup Q^{(1)})), \tag{218}$$

$$\bar{K}_{\mathbb{P}}^{\mathbb{V}(E)} := \bigcup_{(q_1,(q,\check{a}),p_1)\in\bar{T}_{\mathbb{P}}^{\mathbb{V}(E)}} \begin{aligned}[t] &\{((\tau^{(1)})^{-1}(q_1),(q_1,(q,\check{a}),p_1)),\\ &(1_q,(q_1,(q,\check{a}),p_1)),\\ &\{((\tau^{(1)})^{-1}(q),(q_1,(q,\check{a}),p_1)),\\ &((q_1,(q,\check{a}),p_1),(\tau^{(1)})^{-1}(p_1)),\\ &((q_1,(q,\check{a}),p_1),\check{0})\}\subset\end{aligned}$$
$$((Q_{\mathbb{V}}^{(1)}\cup(Z_{\mathbb{P}}(E_{\mathbb{P}})\cup\{\check{0}\})\cup Q^{(1)})\times\bar{T}_{\mathbb{P}}^{\mathbb{V}(E)})\cup$$
$$(\bar{T}_{\mathbb{P}}^{\mathbb{V}(E)}\times(Q_{\mathbb{V}}^{(1)}\cup(Z_{\mathbb{P}}(E_{\mathbb{P}})\cup\{\check{0}\}))) \quad \text{and} \tag{219}$$

$$\begin{aligned}\tilde{\tilde{K}}_{\mathbb{P}}^{\mathbb{V}(E)} := &\bigcup_{(q_1,\breve{a},p_1)\in\tilde{\tilde{T}}_{\mathbb{P}}^{\mathbb{V}(E)}} \\ &\qquad\{((\tau^{(1)})^{-1}(q_1),(q_1,\breve{a},p_1)), \\ &\qquad (0,(q_1,\breve{a},p_1)), \\ &\qquad ((q_1,\breve{a},p_1),(\tau^{(1)})^{-1}(p_1)), \\ &\qquad ((q_1,\breve{a},p_1),\breve{0})\} \subset \\ &((Q_{\mathbb{V}}^{(1)} \dot\cup (Z_{\mathbb{P}}(E_{\mathbb{P}}) \dot\cup \{\breve{0}\})) \times \tilde{\tilde{T}}_{\mathbb{P}}^{\mathbb{V}(E)}) \dot\cup \\ &(\tilde{\tilde{T}}_{\mathbb{P}}^{\mathbb{V}(E)} \times (Q_{\mathbb{V}}^{(1)} \dot\cup (Z_{\mathbb{P}}(E_{\mathbb{P}}) \dot\cup \{\breve{0}\}))). \end{aligned} \tag{220}$$

Let now the sets $\Omega_{\mathbb{P}}^{\mathbb{V}}$, $\mathcal{O}_{\mathbb{P}}^{\mathbb{V}}$ and $\mathcal{E}_{\mathbb{P}}^{\mathbb{V}}(M)$ as well as the functions $\mathcal{I}_{\mathbb{P}}^{\mathbb{V}}$ and $\mathcal{F}_{\mathbb{P}}^{\mathbb{V}}$ be defined corresponding to (155), (156) and (157). By induction on the length of occurrence sequences $o \in (\mathcal{I}_{\mathbb{P}}^{\mathbb{V}})^{-1}(\iota_{\mathbb{P}}^{\mathbb{V}}((q_{\mathbb{V}_0}, q_{\mathbb{V}_0}, (0,0,0))))$ it can be shown that

$$\begin{aligned}&\sum_{x\in Q_{\mathbb{V}}^{(1)}} M(x) = \sum_{x\in Q_{\mathbb{V}}^{(2)}} M(x) = \sum_{x\in Z_{\mathbb{P}}(E_{\mathbb{P}})\dot\cup\{\breve{0}\}} M(x) = 1 \\ &\text{for each } M \in \mathcal{E}_{\mathbb{P}}^{\mathbb{V}}(\iota_{\mathbb{P}}^{\mathbb{V}}((q_{\mathbb{V}_0}, q_{\mathbb{V}_0}, (0,0,0))).\end{aligned} \tag{221}$$

Therefore the function $\zeta_{\mathbb{P}}^{\mathbb{V}(3)} : \mathcal{E}_{\mathbb{P}}^{\mathbb{V}}(\iota_{\mathbb{P}}^{\mathbb{V}}((q_{\mathbb{V}_0}, q_{\mathbb{V}_0}, (0,0,0))) \to \mathbb{N}_0^Q$ is well defined for each $M \in \mathcal{E}_{\mathbb{P}}^{\mathbb{V}}(\iota_{\mathbb{P}}^{\mathbb{V}}((q_{\mathbb{V}_0}, q_{\mathbb{V}_0}, (0,0,0)))$ by

$$\begin{aligned}&\zeta_{\mathbb{P}}^{\mathbb{V}(3)}(M) := 0 \text{ if } M(\breve{0}) = 1 \text{ and} \\ &\zeta_{\mathbb{P}}^{\mathbb{V}(3)}(M) := f \text{ if } M(f) = 1 \text{ for } f \in Z_{\mathbb{P}}(E_{\mathbb{P}}).\end{aligned} \tag{222}$$

For $i \in \{1,2\}$ let the functions $\zeta_{\mathbb{P}}^{\mathbb{V}(i)} : \mathcal{E}_{\mathbb{P}}^{\mathbb{V}}(\iota_{\mathbb{P}}^{\mathbb{V}}((q_{\mathbb{V}_0}, q_{\mathbb{V}_0}, (0,0,0))) \to \mathbb{N}_0^Q$ for each $M \in \mathcal{E}_{\mathbb{P}}^{\mathbb{V}}(\iota_{\mathbb{P}}^{\mathbb{V}}((q_{\mathbb{V}_0}, q_{\mathbb{V}_0}, (0,0,0)))$ be defined by

$$\begin{aligned}&\zeta_{\mathbb{P}}^{\mathbb{V}(1)}(M)(q) := M((\tau^{(1)})^{-1}(q)) \text{ and} \\ &\zeta_{\mathbb{P}}^{\mathbb{V}(2)}(M)(q) := M((\tau^{(2)})^{-1}(q)) \text{ for each } q \in Q.\end{aligned} \tag{223}$$

An induction as for (221) proves

$$\begin{aligned}&\zeta_{\mathbb{P}}^{\mathbb{V}(1)}(M) = \zeta_{\mathbb{P}}^{\mathbb{V}(2)}(M) + \zeta_{\mathbb{P}}^{\mathbb{V}(3)}(M), \\ &\zeta_{\mathbb{P}}^{\mathbb{V}(1)}(M) \in Z_{\mathbb{P}}(A_{\mathbb{P}}) \text{ and } \zeta_{\mathbb{P}}^{\mathbb{V}(2)}(M) \in Z_{\mathbb{P}}(A_{\mathbb{P}}) \\ &\text{for each } M \in \mathcal{E}_{\mathbb{P}}^{\mathbb{V}}(\iota_{\mathbb{P}}^{\mathbb{V}}((q_{\mathbb{V}_0}, q_{\mathbb{V}_0}, (0,0,0))).\end{aligned} \tag{224}$$

To formulate the main theorem about the simulation of $\mathbb{W}_{\mathbb{P}}^{\mathbb{V}}$ by $N_{\mathbb{P}}^{\mathbb{V}}$ let the mapping

$$\begin{aligned}
&\sigma_{\mathbb{P}}^{\mathbb{V}()} : \Psi_{\mathbb{P}}^{\mathbb{V}} \to (2^{Q_{\mathbb{V}} \times Q_{\mathbb{V}} \times \sqcup_{\mathbb{P}}^{\sigma}} \setminus \{\emptyset\}) \cup (2^{Q_{\mathbb{V}} \times \sqcup_{\mathbb{P}}^{\sigma}} \setminus \{\emptyset\}) \text{ be defined by}\\
&\sigma_{\mathbb{P}}^{\mathbb{V}()}(((y_1, y_2), x, (y_1', y_2'))) := \{y_1\} \times \{y_2\} \times \sigma_{\mathbb{P}}^{()} \text{ for}\\
&((y_1, y_2), x, (y_1', y_2')) \in \Psi_{\mathbb{P}}^{\mathbb{V}} \cap (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) \times \sqcup_{\mathbb{P}}^{(S)} \times (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) \text{ and}\\
&\sigma_{\mathbb{P}}^{\mathbb{V}()}(((y_1, y_2), x, (y_1', y_2'))) := \{y_1\} \times \sigma_{\mathbb{P}}^{()} \text{ for}\\
&((y_1, y_2), x, (y_1', y_2')) \in \Psi_{\mathbb{P}}^{\mathbb{V}} \cap (Q_{\mathbb{V}} \times Q_{\mathbb{V}}) \times \sqcup_{\mathbb{P}}^{(E)} \times (Q_{\mathbb{V}} \times Q_{\mathbb{V}}).
\end{aligned} \tag{225}$$

Now, together with (224) and (221) an induction on the length of $w \in W_{\mathbb{P}}^{\mathbb{V}}$ proves

Theorem 19.

For each $o = o_1 ... o_{|o|} \in (\mathbb{N}_0^{R_{\mathbb{P}}^{\mathbb{V}}} \times T_{\mathbb{P}}^{\mathbb{V}} \times \mathbb{N}_0^{R_{\mathbb{P}}^{\mathbb{V}}})^+$ with $o_i \in \mathbb{N}_0^{R_{\mathbb{P}}^{\mathbb{V}}} \times T_{\mathbb{P}}^{\mathbb{V}} \times \mathbb{N}_0^{R_{\mathbb{P}}^{\mathbb{V}}}$ for $1 \le i \le |o|$ holds $o \in (\mathcal{I}_{\mathbb{P}}^{\mathbb{V}})^{-1}(\iota_{\mathbb{P}}^{\mathbb{V}}(q_{\mathbb{P}_0}^{\mathbb{V}}))$, iff there exists $w \in W_{\mathbb{P}}^{\mathbb{V}}$ with $|w| = |o|$ such that for $1 \le i \le |o|$ holds: $o_i = (\iota_{\mathbb{P}}^{\mathbb{V}}(\lambda_{\mathbb{P}}^{\mathbb{V}}(q_{\mathbb{P}_0}^{\mathbb{V}}, w_{i-1}')), t_i, \iota_{\mathbb{P}}^{\mathbb{V}}(\lambda_{\mathbb{P}}^{\mathbb{V}}(q_{\mathbb{P}_0}^{\mathbb{V}}, w_i')))$ with $w_j' \in \mathrm{pre}(w)$, $|w_j'| = j$ for $0 \le j \le |o|$, and $t_i \in \chi_{\mathbb{P}}^{\mathbb{V}}(\sigma_{\mathbb{P}}^{\mathbb{V}()}(w_i))$, where $w = w_1 \dots w_{|o|}$ and $w_i \in \Psi_{\mathbb{P}}^{\mathbb{V}}$ for $1 \le i \le |o|$.

Theorem 19 implies

$$\iota_{\mathbb{P}}^{\mathbb{V}}(\lambda_{\mathbb{P}}^{\mathbb{V}}(q_{\mathbb{P}_0}^{\mathbb{V}}, W_{\mathbb{P}}^{\mathbb{V}})) = \mathcal{E}_{\mathbb{P}}^{\mathbb{V}}(\iota_{\mathbb{P}}^{\mathbb{V}}(q_{\mathbb{P}_0}^{\mathbb{V}})). \tag{226}$$

As the reachability problem of Petri nets is decidable [12], [13], by (194) and (226) follows

Corollary 10.
$\mathrm{SP}(P \cup \{\varepsilon\}, V)$ *is decidable for regular languages P and V.*

The decidability of $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ essentially depends on the decidability of the Petri net reachability problem. In [12] this decidability result is annotated as double complex: in the proof and in the algorithm. For practical applications it is therefore important, to have simpler sufficient conditions for $\mathrm{SP}(P \cup \{\varepsilon\}, V)$, as demonstrated in Example 10, Example 11 and in Example 13.

Appendix

A Shuffle Projection in Terms of Shuffle Factors

The *shuffle product* $U \amalg V$ [1] for languages U and V can be defined in terms of the homomorphisms τ_n^I and Θ^I.

Definition 28.
For $U, V \subset \Sigma^*$ *the* shuffle product $U \amalg V \subset \Sigma^*$ *is defined by*

$$U \amalg V := \Theta^{\{1,2\}}[(\tau_1^{\{1,2\}})^{-1}(U) \cap (\tau_2^{\{1,2\}})^{-1}(V)].$$

It is easy to see that

$$\begin{aligned}
&\amalg \text{ is commutative, } \{w\} = \{w\} \amalg \{\varepsilon\} \text{ for } w \in \Sigma^*,\\
&|w| = |u| + |v| \text{ for } w \in \{u\} \amalg \{v\} \text{ and } u, v \in \Sigma^*,\\
&\mathrm{pre}(U \amalg V) = \mathrm{pre}(U) \amalg \mathrm{pre}(V), \text{ and}\\
&U \amalg V = \bigcup_{u \in U, v \in V} \{u\} \amalg \{v\} \text{ for } U, V \subset \Sigma^*.\\
&\text{By Lemma 4}\\
&\{1,2\} \text{ can be replaced by any set } S \text{ with } \#(S) = 2. \qquad (227)
\end{aligned}$$

The following lemma is the key to a relation between shuffle products and shuffle projection.

Lemma 12.
Let $P \subset \Sigma^*$. *Then* $w \in \{u\} \amalg \{v\}$ *for* $u, v \in P^{\sqcup}$, *iff there exist*

$$\begin{aligned}
&x \in \bigcap_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\}) \text{ and } K \subset \mathbb{N} \text{ with } w = \Theta^{\mathbb{N}}(x) \in P^{\sqcup},\\
&u = \Theta^K(\Pi_K^{\mathbb{N}}(x)) \text{ and } v = \Theta^{\mathbb{N} \setminus K}(\Pi_{\mathbb{N} \setminus K}^{\mathbb{N}}(x)).
\end{aligned}$$

Proof.
Let $x \in \bigcap_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})$ and $K \subset \mathbb{N}$, then $w := \Theta^{\mathbb{N}}(x) \in P^{\sqcup}$ and by Lemma 2 $u := \Theta^K(\Pi_K^{\mathbb{N}}(x)) \in P^{\sqcup}$ and $v := \Theta^{\mathbb{N} \setminus K}(\Pi_{\mathbb{N} \setminus K}^{\mathbb{N}}(x)) \in P^{\sqcup}$.
Let $\omega_K : \Sigma_{\mathbb{N}}^* \to \Sigma_{\{1,2\}}^*$ be defined by $\omega_K(a) := (\tau_1^{\{1\}})^{-1}(\Theta^K(a))$ for $a \in \Sigma_K$ and $\omega_K(a) := (\tau_2^{\{2\}})^{-1}(\Theta^{\mathbb{N} \setminus K}(a))$ for $a \in \Sigma_{\mathbb{N} \setminus K}$, then $w = \Theta^{\mathbb{N}}(x) = \Theta^{\{1,2\}}(\omega_K(x))$ and $\omega_K(x) \in (\tau_1^{\{1,2\}})^{-1}(\{u\}) \cap (\tau_2^{\{1,2\}})^{-1}(\{v\})$. This implies $w \in \{u\} \amalg \{v\}$.

Let now $u, v \in P^{\sqcup}$ and $w \in \{u\} \amalg \{v\}$. Then there exist
$\breve{w} \in (\tau_1^{\{1,2\}})^{-1}(\{u\}) \cap (\tau_2^{\{1,2\}})^{-1}(\{v\})$ with $\Theta^{\{1,2\}}(\breve{w}) = w$,
$\breve{u} \in \bigcap_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})$ with $\Theta^{\mathbb{N}}(\breve{u}) = u$ and

$\breve{v} \in \bigcap_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})$ with $\Theta^{\mathbb{N}}(\breve{v}) = v$.

Then, by "combining the structures" of $\breve{w}$, $\breve{u}$ and $\breve{v}$ there exists

$$\begin{aligned}
&\breve{x} \in \bigcap_{n \in \mathbb{N}\times\{1,2\}} (\tau_n^{\mathbb{N}\times\{1,2\}})^{-1}(P \cup \{\varepsilon\}) \text{ with } \Theta_{\{1,2\}}^{\mathbb{N}\times\{1,2\}}(\breve{x}) = \breve{w}, \\
&\Theta_{\mathbb{N}}^{\mathbb{N}\times\{1\}}(\Pi_{\mathbb{N}\times\{1\}}^{\mathbb{N}\times\{1,2\}}(\breve{x})) = \breve{u},\ \Theta_{\mathbb{N}}^{\mathbb{N}\times\{2\}}(\Pi_{\mathbb{N}\times\{2\}}^{\mathbb{N}\times\{1,2\}}(\breve{x})) = \breve{v}, \\
&|\Pi_{\mathbb{N}\times\{1\}}^{\mathbb{N}\times\{1,2\}}(x')| = |\tau_1^{\{1,2\}}(w')| \text{ and } |\Pi_{\mathbb{N}\times\{2\}}^{\mathbb{N}\times\{1,2\}}(x')| = |\tau_2^{\{1,2\}}(w')| \\
&\text{for each } x' \in \operatorname{pre}(\breve{x}) \text{ and } w' \in \operatorname{pre}(\breve{w}) \text{ with } |x'| = |w'|.
\end{aligned} \tag{228}$$

This implies $w = \Theta^{\mathbb{N}\times\{1,2\}}(\breve{x})$,

$$u = \Theta^{\mathbb{N}\times\{1\}}(\Pi_{\mathbb{N}\times\{1\}}^{\mathbb{N}\times\{1,2\}}(\breve{x})) \text{ and } v = \Theta^{\mathbb{N}\times\{2\}}(\Pi_{\mathbb{N}\times\{2\}}^{\mathbb{N}\times\{1,2\}}(\breve{x})). \tag{229}$$

Each bijection $\iota : N \to N'$ defines an isomorphism $\iota_{N'}^{N} : \Sigma_N^* \to \Sigma_{N'}^*$ by $\iota_{N'}^{N} := (\tau_{\iota(i)}^{\{\iota(i)\}})^{-1}(\tau_i^{\{i\}}(a))$ for $a \in \Sigma_{\{i\}}$ and $i \in N$. Then it is easy to see [11] that

$$\begin{aligned}
&\iota_{N'}^{N}(x) \in \bigcap_{t \in N'} (\tau_t^{N'})^{-1}(P \cup \{\varepsilon\}) \text{ and } \Theta^{K}(\Pi_K^{N}(x)) = \Theta^{\iota(K)}(\Pi_{\iota(K)}^{N'}(\iota_{N'}^{N}(x))) \\
&\text{for } x \in \bigcap_{t \in N} (\tau_t^{N})^{-1}(P \cup \{\varepsilon\}) \text{ and } K \subset N.
\end{aligned} \tag{230}$$

Applying (230) with $N = \mathbb{N}\times\{1,2\}$ and $N' = \mathbb{N}$ to (228) and (229) completes the proof of the lemma.

Moreover, the second part of this proof shows

Corollary 11.
Let $P \subset \Sigma^*$. Then $w \in \{u\}⧢\{v\}$ for $u, v \in P^{\sqcup}$, iff there exist

$$\begin{aligned}
&x \in \bigcap_{t \in \mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\}) \textit{ and } K \subset \mathbb{N} \textit{ with } \#(K) = \#(\mathbb{N}\setminus K) = \#(\mathbb{N}), \\
&w = \Theta^{\mathbb{N}}(x) \in P^{\sqcup},\ u = \Theta^{K}(\Pi_K^{\mathbb{N}}(x)) \textit{ and } v = \Theta^{\mathbb{N}\setminus K}(\Pi_{\mathbb{N}\setminus K}^{\mathbb{N}}(x)).
\end{aligned}$$

Lemma 12 and corollary 11 motivates

Definition 29.
For $P \subset \Sigma^*$ let $\mathrm{SF}_P : 2^{P^{\sqcup}} \to 2^{P^{\sqcup}}$ be defined by

$$\mathrm{SF}_P(M) := \{u \in P^{\sqcup} |\ \textit{there exist } w \in M \textit{ and } v \in P^{\sqcup} \textit{ such that } w \in \{u\}⧢\{v\}\}$$

for $M \subset P^{\sqcup}$. The elements of $\mathrm{SF}_P(M)$ are called shuffle factors *of M.*

It is an immediate consequence of this definition that

$$\begin{aligned}
&M \subset \mathrm{SF}_P(M),\ \mathrm{SF}_P(M) = \bigcup_{w \in M} \mathrm{SF}_P(\{w\}) \\
&\text{and therefore } \mathrm{SF}_P(U) \subset \mathrm{SF}_P(M) \text{ for } U \subset M \subset P^{\sqcup}.
\end{aligned} \tag{231}$$

Theorem 20.
Let $P,V \subset \Sigma^*$*. Then* $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ *iff* $\mathrm{SF}_P(P^{\sqcup\!\sqcup} \cap V) \subset P^{\sqcup\!\sqcup} \cap V$.

Proof.
By Corollary 11 $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ implies $\mathrm{SF}_P(P^{\sqcup\!\sqcup} \cap V) \subset P^{\sqcup\!\sqcup} \cap V$, which implies $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ on account of Lemma 12.

Remark. This proof shows that in Definition 7 the restriction $K \neq \emptyset$ can be omitted, or K can also be restricted by $\#(K) = \#(\mathbb{N} \setminus K) = \#(\mathbb{N})$.

Additionally to commutativity also associativity of Ш is well known, see for example [5]. Because of $U Ш V = \bigcup\limits_{u \in U, v \in V} \{u\} Ш \{v\}$, the following lemma is sufficient for its proof.

Lemma 13.
Let $u,v,w \in \Sigma^*$*. Then*

$$(\{u\} Ш \{v\}) Ш \{w\} =$$
$$\Theta^{\{1,2,3\}}[(\tau_1^{\{1,2,3\}})^{-1}(\{u\}) \cap (\tau_2^{\{1,2,3\}})^{-1}(\{v\}) \cap (\tau_3^{\{1,2,3\}})^{-1}(\{w\})] =$$
$$\{u\} Ш (\{v\} Ш \{w\}).$$

Proof. $x \in (\{u\} Ш \{v\}) Ш \{w\}$, iff

$$\text{there exists } y \in \{u\} Ш \{v\} \text{ with } x \in \{y\} Ш \{w\}. \tag{232}$$

(232) is equivalent to:

$$\begin{aligned}&\text{There exist } \breve{y} \in (\tau_1^{\{1,2\}})^{-1}(\{u\}) \cap (\tau_2^{\{1,2\}})^{-1}(\{v\}) \text{ and}\\ &\breve{x} \in (\tau_{\{1,2\}}^{\{\{1,2\},3\}})^{-1}(\{y\}) \cap (\tau_3^{\{\{1,2\},3\}})^{-1}(\{w\}) \text{ with}\\ &y = \Theta^{\{1,2\}}(\breve{y}) \text{ and } x = \Theta^{\{\{1,2\},3\}}(\breve{x}).\end{aligned} \tag{233}$$

(233) is equivalent to:

$$\begin{aligned}&\text{There exists } \breve{z} \in (\tau_1^{\{1,2,3\}})^{-1}(\{u\}) \cap (\tau_2^{\{1,2,3\}})^{-1}(\{v\}) \cap (\tau_3^{\{1,2,3\}})^{-1}(\{w\}) \text{ with}\\ &\Pi_{\{1,2\}}^{\{1,2,3\}}(\breve{z}) = \breve{y},\ \Theta^{\{1,2,3\}}(\breve{z}) = x,\\ &|\Pi_{\{1,2\}}^{\{1,2,3\}}(z')| = |\tau_{\{1,2\}}^{\{\{1,2\},3\}}(x')| \text{ and } |\Pi_{\{3\}}^{\{1,2,3\}}(z')| = |\tau_3^{\{\{1,2\},3\}}(x')|\\ &\text{for each } z' \in \mathrm{pre}(\breve{z}) \text{ and } x' \in \mathrm{pre}(\breve{x}) \text{ with } |z'| = |x'|.\end{aligned} \tag{234}$$

where $\breve{z}$ result by "combining the structures" of $\breve{y}$ and $\breve{x}$. (232) - (234) proves the first equation of the lemma. The second equation can be shown by an analogous argument.

Lemma 13 shows:

$$u \in \mathrm{SF}_P(\{w\}) \text{ and } x \in \mathrm{SF}_P(\{u\}) \text{ implies } x \in \mathrm{SF}_P(\{w\}) \text{ for each } w \in P^{\sqcup\!\sqcup}.$$

Therefore $\mathrm{SF}_P(\mathrm{SF}_P(M)) \subset \mathrm{SF}_P(M)$ for each $M \subset P^{ш}$, which by (231) implies

$$\mathrm{SF}_P(\mathrm{SF}_P(M)) = \mathrm{SF}_P(M) \text{for each M} \subset P^{ш}. \tag{235}$$

On account of (231) and (235) SF_P is a closure operator [3]. For $P, V \subset \Sigma^*$ and $M \subset P^{ш}$, by Theorem 20 $\mathrm{SF}_P(M)$ is the smallest V with $X \subset V$ and $\mathrm{SP}(P \cup \{\varepsilon\}, V)$. On account of (6) holds

$$\mathrm{SF}_P(M) = \bigcap_{\substack{M \subset V \subset \Sigma^* \\ \mathrm{SP}(P \cup \{\varepsilon\}, V)}} V. \tag{236}$$

For $P \subset \Sigma^*$, SF_P is a generalization of $\mathcal{C}_\Sigma$, where

$$\mathcal{C}_\Sigma(X) := \{u \in \Sigma^* | \text{ there exist } n \in \mathbb{N} \text{ and } u_i, v_i \in \Sigma^* \text{ for } 1 \leq i \leq n \text{ such that } u = u_1 ... u_n \text{ and } u_1 v_1 ... u_n v_n \in X\} = \mathrm{SF}_\Sigma(X)$$

for $X \subset \Sigma^{ш} = \Sigma^*$ [4], which is called the *downward closure* of X.

In preparation for the next section we show the following

Lemma 14.

$$\{ua\}Ш\{vb\} = (\{u\}Ш\{vb\})a \cup (\{ua\}Ш\{v\})b \quad \textit{for } u, v \in \Sigma^* \textit{ and } a, b \in \Sigma.$$

Proof.
On account of (227) $\{ua\}Ш\{vb\} \subset \Sigma^+$, and therefore

$$\begin{aligned}
&\{ua\}Ш\{vb\} = \Theta^{\{1,2\}}[(\tau_1^{\{1,2\}})^{-1}(\{ua\}) \cap (\tau_2^{\{1,2\}})^{-1}(\{vb\})] = \\
&\Theta^{\{1,2\}}[(\tau_1^{\{1,2\}})^{-1}(\{ua\}) \cap (\tau_2^{\{1,2\}})^{-1}(\{vb\}) \cap \Sigma^*_{\{1,2\}}\Sigma_{\{1\}}] \cup \\
&\Theta^{\{1,2\}}[(\tau_1^{\{1,2\}})^{-1}(\{ua\}) \cap (\tau_2^{\{1,2\}})^{-1}(\{vb\}) \cap \Sigma^*_{\{1,2\}}\Sigma_{\{2\}}] = \\
&\Theta^{\{1,2\}}[(\tau_1^{\{1,2\}})^{-1}(\{u\}) \cap (\tau_2^{\{1,2\}})^{-1}(\{vb\})]a \cup \\
&\Theta^{\{1,2\}}[(\tau_1^{\{1,2\}})^{-1}(\{ua\}) \cap (\tau_2^{\{1,2\}})^{-1}(\{v\})]b = \\
&(\{u\}Ш\{vb\})a \cup (\{ua\}Ш\{v\})b.
\end{aligned}$$

The properties of (227) and Lemma 14 completely characterize Ш. It is well known that

$$\begin{aligned}
&\{w\} = \{w\}Ш\{\varepsilon\} = \{\varepsilon\}Ш\{w\} \text{ for } w \in \Sigma^*, \\
&\{ua\}Ш\{vb\} = (\{u\}Ш\{vb\})a \cup (\{ua\}Ш\{v\})b \quad \text{for } u, v \in \Sigma^* \text{ and } a, b \in \Sigma, \text{ and} \\
&UШV = \bigcup_{u \in U, v \in V} \{u\}Ш\{v\} \quad \text{for } U, V \subset \Sigma^*
\end{aligned} \tag{237}$$

inductively defines Ш, see for example [7].

Lemma 12 shows

$$\text{For each } \varepsilon \neq w \in P^{\sqcup\!\sqcup} \text{ there exist } \varepsilon \neq e \in P \text{ and } v \in P^{\sqcup\!\sqcup} \text{ with } w \in \{e\} Ш \{v\}. \tag{238}$$

(238) together with Lemma 12 implies the following well known inductive definition of $P^{\sqcup\!\sqcup}$, see for example [7]:

$$P^{\sqcup\!\sqcup} = \bigcup_{n\in\mathbb{N}} P^{(\sqcup\!\sqcup,n)} \text{ where}$$
$$P^{(\sqcup\!\sqcup,1)} := P \cup \{\varepsilon\} \text{ and } P^{(\sqcup\!\sqcup,n+1)} := P^{(\sqcup\!\sqcup,n)} Ш (P \cup \{\varepsilon\}) \text{ for } n \in \mathbb{N}. \tag{239}$$

(239) motivates

Definition 30.
For $P \subset \Sigma^*$ and $n \in \mathbb{N}$ let $\mathrm{SF}_P^{(n)} : 2^{P^{\sqcup\!\sqcup}} \to 2^{P^{\sqcup\!\sqcup}}$ be defined by $\mathrm{SF}_P^{(n)}(M) :=$

$$\{u \in P^{\sqcup\!\sqcup} \mid \textit{there exist } w \in M \textit{ and } v \in P^{(\sqcup\!\sqcup,n)} \textit{ such that } w \in \{u\} Ш \{v\}\}$$

for $M \subset P^{\sqcup\!\sqcup}$.

It is an immediate consequence of this definition that

$$\mathrm{SF}_P^{(n)}(M) = \bigcup_{w\in M} \mathrm{SF}_P^{(n)}(\{w\}) \text{ and therefore}$$
$$\mathrm{SF}_P^{(n)}(U) \subset \mathrm{SF}_P^{(n)}(M) \text{ for } U \subset M \subset P^{\sqcup\!\sqcup} \text{ and } n \in \mathbb{N}. \tag{240}$$

Since $\{\varepsilon\} \subset P^{(\sqcup\!\sqcup,n)} \subset P^{(\sqcup\!\sqcup,n+1)}$ for $n \in \mathbb{N}$, (239) implies

$$M \subset \mathrm{SF}_P^{(n)}(M) \subset \mathrm{SF}_P^{(n+1)}(M) \text{ for } n \in \mathbb{N},$$
$$\text{and } \mathrm{SF}_P(M) = \bigcup_{n\in\mathbb{N}} \mathrm{SF}_P^{(n)}(M) \text{ for } M \subset P^{\sqcup\!\sqcup}. \tag{241}$$

The iterative definition of $P^{(\sqcup\!\sqcup,n)}$ together with the commutativity and associativity of Ш shows:

$$\mathrm{SF}_P^{(n+1)}(M) = \mathrm{SF}_P^{(1)}(\mathrm{SF}_P^{(n)}(M)) = \mathrm{SF}_P^{(n)}(\mathrm{SF}_P^{(1)}(M))$$
$$\text{for } M \subset P^{\sqcup\!\sqcup} \text{ and } n \in \mathbb{N}. \tag{242}$$

For $M \subset P^{\sqcup\!\sqcup}$ (242) by induction implies

$$\mathrm{SF}_P^{(1)}(M) \subset M \text{ iff } \mathrm{SF}_P^{(n)}(M) \subset M \text{ for each } n \in \mathbb{N}.$$

Therefore, by (241) and Theorem 20 holds

Corollary 12.
Let $P, V \subset \Sigma^*$. Then $\mathrm{SP}(P \cup \{\varepsilon\}, V)$ iff $\mathrm{SF}_P^{(1)}(P^{\sqcup\!\sqcup} \cap V) \subset P^{\sqcup\!\sqcup} \cap V$.

By Lemma 12 Corollary 12 is a reformulation of Theorem 6.

B Shuffled Runs of Computations in S-Automata

To represent $\mathrm{SF}^{(1)}_{\mathrm{pre}(P)}$ and $\mathrm{SF}_{\mathrm{pre}(P)}$ for $\emptyset \neq P \subset \Sigma^*$ in terms of computations in S-automata, now a kind of shuffle product will be defined on $2^{A_{\mathbb{P}}}$. Guideline for this definition is (237). Preparatively let $\mathbb{P}$ and $A_{\mathbb{P}}$ be defined as in section 4, and let

$$\begin{aligned}
&\{c\}\text{Ш}^{(\mathbb{P})}\{\varepsilon\} := \{\varepsilon\}\text{Ш}^{(\mathbb{P})}\{c\} := \{c\} \text{ for } c \in A_{\mathbb{P}}, \text{ and}\\
&\{c(f,a,f')\}\text{Ш}^{(\mathbb{P})}\{d(g,b,g')\} :=\\
&(\{c\}\text{Ш}^{(\mathbb{P})}\{d(g,b,g')\})(f+g',a,f'+g') \cup (\{c(f,a,f')\}\text{Ш}^{(\mathbb{P})}\{d\})(f'+g,b,f'+g')\\
&\text{for } c(f,a,f'), d(g,b,g') \in A_{\mathbb{P}} \text{ with } (f,a,f'),(g,b,g') \in \sqcup_{\mathbb{P}}.
\end{aligned}$$

Then (132) and induction show

$$\begin{aligned}
&\{x\}\text{Ш}^{(\mathbb{P})}\{y\} = \{y\}\text{Ш}^{(\mathbb{P})}\{x\},\ \{x\}\text{Ш}^{(\mathbb{P})}\{y\} \subset A_{\mathbb{P}},\\
&|c| = |x| + |y| \text{ and } Z_{\mathbb{P}}(c) = Z_{\mathbb{P}}(x) + Z_{\mathbb{P}}(y)\\
&\text{for } x,y \in A_{\mathbb{P}} \text{ and } c \in \{x\}\text{Ш}^{(\mathbb{P})}\{y\}.
\end{aligned} \tag{243}$$

Definition 31.
Using (243), let the commutativ operation $\text{Ш}^{(\mathbb{P})} : 2^{A_{\mathbb{P}}} \times 2^{A_{\mathbb{P}}} \to 2^{A_{\mathbb{P}}}$ in infix notation be inductively defined by

$$\begin{aligned}
&\{c\}\text{Ш}^{(\mathbb{P})}\{\varepsilon\} := \{\varepsilon\}\text{Ш}^{(\mathbb{P})}\{c\} := \{c\} \textit{ for } c \in A_{\mathbb{P}},\\
&\{c(f,a,f')\}\text{Ш}^{(\mathbb{P})}\{d(g,b,g')\} :=\\
&(\{c\}\text{Ш}^{(\mathbb{P})}\{d(g,b,g')\})(f+g',a,f'+g') \cup (\{c(f,a,f')\}\text{Ш}^{(\mathbb{P})}\{d\})(f'+g,b,f'+g')\\
&\textit{for } c(f,a,f'), d(g,b,g') \in A_{\mathbb{P}} \textit{ with } (f,a,f'),(g,b,g') \in \sqcup_{\mathbb{P}}, \textit{ and}\\
&X\text{Ш}^{(\mathbb{P})}Y = \bigcup_{x\in X, y\in Y} \{x\}\text{Ш}^{(\mathbb{P})}\{y\} \textit{ for } X,Y \subset A_{\mathbb{P}}.
\end{aligned}$$

$X\text{Ш}^{(\mathbb{P})}Y$ is called the shuffled runs *of X and Y.*

The name *shuffled runs* is justified by the relation to section 5, as will be demonstrated in the last theorem of this section.

Definition 31 allows to transfer Definition 29 to $A_{\mathbb{P}}$:

Definition 32.
Let $\mathrm{SRF}_{\mathbb{P}}: 2^{A_{\mathbb{P}}} \to 2^{A_{\mathbb{P}}}$ *and* $\mathrm{SRF}^{(1)}_{\mathbb{P}}: 2^{A_{\mathbb{P}}} \to 2^{A_{\mathbb{P}}}$ *be defined by*

$\mathrm{SRF}_{\mathbb{P}}(M) :=$

$\{u \in A_{\mathbb{P}} |$ *there exist* $w \in M$ *and* $v \in A_{\mathbb{P}}$ *such that* $w \in \{u\} Ш^{(\mathbb{P})} \{v\}\}$

and

$\mathrm{SRF}^{(1)}_{\mathbb{P}}(M) :=$

$\{u \in A_{\mathbb{P}} |$ *there exist* $w \in M$ *and* $e \in E_{\mathbb{P}}$ *such that* $w \in \{u\} Ш^{(\mathbb{P})} \{e\}\}$

for $M \subset A_{\mathbb{P}}$.

The following two lemmas are the key to express $\mathrm{SF}_{\mathrm{pre}(P)}$ resp. $\mathrm{SF}^{(1)}_{\mathrm{pre}(P)}$ by $\mathrm{SRF}_{\mathbb{P}}$ resp. $\mathrm{SRF}^{(1)}_{\mathbb{P}}$.

Lemma 15.
For $c, d \in A_{\mathbb{P}}$ *and* $x \in \{c\} Ш^{(\mathbb{P})} \{d\}$ *holds* $\alpha_{\mathbb{P}}(x) \in \{\alpha_{\mathbb{P}}(c)\} Ш \{\alpha_{\mathbb{P}}(d)\}$.

Proof (by induction).
Induction base
Let $c = \varepsilon$ or $d = \varepsilon$. On account of commutativity of $Ш^{(\mathbb{P})}$ let $d = \varepsilon$. Then $x = c$ and $\alpha_{\mathbb{P}}(d) = \varepsilon$, which implies $\alpha_{\mathbb{P}}(x) \in \{\alpha_{\mathbb{P}}(c)\} Ш \{\alpha_{\mathbb{P}}(d)\}$.
Induction step
$c \neq \varepsilon \neq d$ implies $c = c'(f, a, f')$ and $d = d'(g, b, g')$ with $c', d' \in A_{\mathbb{P}}$ and $(f, a, f'), (g, b, g') \in \sqcup_{\mathbb{P}}$. Therefore $x \in (\{c'\} Ш^{(\mathbb{P})} \{d'(g, b, g')\})(f + g', a, f' + g') \cup (\{c'(f, a, f')\} Ш^{(\mathbb{P})} \{d'\})(f' + g, b, f' + g')$. On account of symmetry it is sufficient to prove the induction step for $x \in (\{c'\} Ш^{(\mathbb{P})} \{d'(g, b, g')\})(f + g', a, f' + g')$, which implies $x = x'(f + g', a, f' + g')$ with $x' \in \{c'\} Ш^{(\mathbb{P})} \{d'(g, b, g')\}$. Now by the induction hypothesis $\alpha_{\mathbb{P}}(x') \in \{\alpha_{\mathbb{P}}(c')\} Ш \{\alpha_{\mathbb{P}}(d')b\}$, and therefore by Lemma 14 $\alpha_{\mathbb{P}}(x) \in (\{\alpha_{\mathbb{P}}(c')\} Ш \{\alpha_{\mathbb{P}}(d')b\})a \subset \{\alpha_{\mathbb{P}}(c')a\} Ш \{\alpha_{\mathbb{P}}(d')b\} = \{\alpha_{\mathbb{P}}(c)\} Ш \{\alpha_{\mathbb{P}}(d)\}$, which completes the proof of Lemma 15.

Lemma 16.
For $u, v \in \mathrm{pre}(P^{\sqcup}) = (\mathrm{pre}(P))^{\sqcup}$, $w \in \{u\} Ш \{v\}$, $c \in \alpha_{\mathbb{P}}^{-1}(u)$ *and* $d \in \alpha_{\mathbb{P}}^{-1}(v)$ *there exists* $x \in \{c\} Ш^{(\mathbb{P})} \{d\}$ *with* $\alpha_{\mathbb{P}}(x) = w$.

Proof (by induction).
Induction base
Let $u = \varepsilon$ or $v = \varepsilon$. On account of commutativity of $Ш$ let $v = \varepsilon$. Then $w = u$ and $d = \varepsilon$, which implies $c \in \{c\} Ш^{(\mathbb{P})} \{d\}$ with $\alpha_{\mathbb{P}}(c) = w$.
Induction step
$u \neq \varepsilon \neq v$ implies $u = u'a$ and $v = v'b$ with $u', v' \in \mathrm{pre}(P^{\sqcup})$ and $a, b \in \Sigma$. Therefore $w \in (\{u'\} Ш \{v'b\})a \cup (\{u'a\} Ш \{v'\})b$, $c = c'(f, a, f')$ and $d = d'(g, b, g')$ with $c' \in \alpha_{\mathbb{P}}^{-1}(u')$, $d' \in \alpha_{\mathbb{P}}^{-1}(v')$, $(f, a, f'), (g, b, g') \in \sqcup_{\mathbb{P}}$, $Z_{\mathbb{P}}(c') = f$ and $Z_{\mathbb{P}}(d') = g$. On account of symmetry it is sufficient to prove the induction step for $w \in (\{u'\} Ш \{v'b\})a$, which implies $w = w'a$ with $w' \in \{u'\} Ш \{v'b\}$. Now by the induction hypothesis there exists $x' \in \{c'\} Ш^{(\mathbb{P})} \{d'(g, b, g')\}$ with $\alpha_{\mathbb{P}}(x') = w'$. Then

$x := x'(f+g', a, f'+g') \in \{c\}Ш^{(\mathbb{P})}\{d\}$ with $\alpha_{\mathbb{P}}(x) = w$, which completes the proof of Lemma 16.

Theorem 21.
$\mathrm{SF}_{\mathrm{pre}(P)}(M) = \alpha_{\mathbb{P}}(\mathrm{SRF}_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(M)))$ for $M \subset \mathrm{pre}(P^{⊔⊔}) = (\mathrm{pre}(P))^{⊔⊔}$.

Proof.
For $u \in \mathrm{SF}_{\mathrm{pre}(P)}(M) \subset \mathrm{pre}(P^{⊔⊔})$ there exist $w \in M$ and $v \in \mathrm{pre}(P^{⊔⊔})$ such that $w \in \{u\}Ш\{v\}$. Now, by Corollary 6 and Lemma 16 there exist $c \in \alpha_{\mathbb{P}}^{-1}(u) \subset A_{\mathbb{P}}$, $d \in \alpha_{\mathbb{P}}^{-1}(v) \subset A_{\mathbb{P}}$, and $x \in \alpha_{\mathbb{P}}^{-1}(w) \subset \alpha_{\mathbb{P}}^{-1}(M)$ with $x \in \{c\}Ш^{(\mathbb{P})}\{d\}$. This implies $c \in \mathrm{SRF}_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(M))$, which proves $u \in \alpha_{\mathbb{P}}(\mathrm{SRF}_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(M)))$.
For $c \in \mathrm{SRF}_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(M)) \subset A_{\mathbb{P}}$ there exist $x \in \alpha_{\mathbb{P}}^{-1}(M)$ and $d \in A_{\mathbb{P}}$ such that $x \in \{c\}Ш^{(\mathbb{P})}\{d\}$. Now, by Corollary 6 and Lemma 15 $\alpha_{\mathbb{P}}(c), \alpha_{\mathbb{P}}(d) \in (\mathrm{pre}(P))^{⊔⊔}$, and $\alpha_{\mathbb{P}}(x) \in \{\alpha_{\mathbb{P}}(c)\}Ш\{\alpha_{\mathbb{P}}(d)\}$, which shows $\alpha_{\mathbb{P}}(c) \in \mathrm{SF}_{\mathrm{pre}(P)}(M)$. This completes the proof of Theorem 21.

The proof of Theorem 21 together with $P^{⊔⊔} = \alpha_{\mathbb{P}}(Z_{\mathbb{P}}^{-1}(0))$ (Corollary 6) and (243) shows

Corollary 13.
$\mathrm{SF}_P(M) = \alpha_{\mathbb{P}}(\mathrm{SRF}_{\mathbb{P}}(\alpha_{\mathbb{P}}^{-1}(M) \cap Z_{\mathbb{P}}^{-1}(0)))$ for $M \subset P^{⊔⊔}$.

Together with $\alpha_{\mathbb{P}}(E_{\mathbb{P}}) = \mathrm{pre}(P)$ and $\alpha_{\mathbb{P}}(E_{\mathbb{P}} \cap Z_{\mathbb{P}}^{-1}(0)) = P$ (59), the proofs of Theorem 21 and Corollary 13 shows

Corollary 14.
$\mathrm{SF}_P^{(1)}(M) = \alpha_{\mathbb{P}}(\mathrm{SRF}_{\mathbb{P}}^{(1)}(\alpha_{\mathbb{P}}^{-1}(M) \cap Z_{\mathbb{P}}^{-1}(0)))$ for $M \subset P^{⊔⊔}$, and
$\mathrm{SF}_{\mathrm{pre}(P)}^{(1)}(M) = \alpha_{\mathbb{P}}(\mathrm{SRF}_{\mathbb{P}}^{(1)}(\alpha_{\mathbb{P}}^{-1}(M)))$ for $M \subset \mathrm{pre}(P^{⊔⊔}) = (\mathrm{pre}(P))^{⊔⊔}$.

Because of $\alpha_{\mathbb{P}}^{-1}((\mathrm{pre}(P))^{⊔⊔} \cap V) = \alpha_{\mathbb{P}}^{-1}(V)$, Corollary 12 and Corollary 14 imply

Corollary 15.
$\mathrm{SP}(\mathrm{pre}(P), V)$ iff $\mathrm{SRF}_{\mathbb{P}}^{(1)}(\alpha_{\mathbb{P}}^{-1}(V)) \subset \alpha_{\mathbb{P}}^{-1}(V)$.

Because of $Z_{\mathbb{P}}^{-1}(0) \subset \alpha_{\mathbb{P}}^{-1}(P^{⊔⊔})$, it holds

$$\begin{aligned}\alpha_{\mathbb{P}}^{-1}(P^{⊔⊔} \cap V) \cap Z_{\mathbb{P}}^{-1}(0) &= \alpha_{\mathbb{P}}^{-1}(P^{⊔⊔}) \cap \alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0) \\ &= \alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0). \end{aligned} \tag{244}$$

(243) implies

$$\mathrm{SRF}_{\mathbb{P}}^{(1)}(\alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0)) \subset Z_{\mathbb{P}}^{-1}(0). \tag{245}$$

By (244) and (245) it holds

$$\begin{aligned}&\mathrm{SRF}_{\mathbb{P}}^{(1)}(\alpha_{\mathbb{P}}^{-1}(P^{⊔⊔} \cap V) \cap Z_{\mathbb{P}}^{-1}(0)) \subset \alpha_{\mathbb{P}}^{-1}(P^{⊔⊔} \cap V) \text{ iff} \\ &\mathrm{SRF}_{\mathbb{P}}^{(1)}(\alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0)) \subset \alpha_{\mathbb{P}}^{-1}(V) \cap Z_{\mathbb{P}}^{-1}(0). \end{aligned} \tag{246}$$

Now, because of (246), Corollary 12 and Corollary 14 imply

Corollary 16.
$\mathrm{SP}(P\cup\{\varepsilon\},V)$ *iff* $\mathrm{SRF}_{\mathbb{P}}^{(1)}(\alpha_{\mathbb{P}}^{-1}(V)\cap Z_{\mathbb{P}}^{-1}(0))\subset\alpha_{\mathbb{P}}^{-1}(V)\cap Z_{\mathbb{P}}^{-1}(0)$.

To show that Corollary 15 and Corollary 16 are equivalent to Corollary 8 and Corollary 9, we prove

Theorem 22. $\mathrm{SRF}_{\mathbb{P}}^{(1)}=\mathcal{R}'_{\mathbb{P}}$.

Proof.
Since $\mathrm{SRF}_{\mathbb{P}}^{(1)}(M)=\bigcup\limits_{x\in M}\mathrm{SRF}_{\mathbb{P}}^{(1)}(\{x\})$ and $\mathcal{R}'_{\mathbb{P}}(M)=\bigcup\limits_{x\in M}\mathcal{R}'_{\mathbb{P}}(\{x\})$ it is sufficient to prove $\mathrm{SRF}_{\mathbb{P}}^{(1)}(\{x\})=\mathcal{R}'_{\mathbb{P}}(\{x\})$ for each $x\in A_{\mathbb{P}}$. For this purpose we show the following:

$$\text{For } x,u\in A_{\mathbb{P}} \text{ and } e\in E_{\mathbb{P}} \text{ holds } x\in\{u\}\text{Ш}^{(\mathbb{P})}\{e\} \text{ iff there exists a shuffled representation } b\in\pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}})\cap\pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}}) \text{ of } x \text{ by } u \text{ and } \check{e}:=\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(e). \tag{247}$$

Because of $\sqcup_{\check{\mathbb{P}}}\cap\sqcup_{\mathbb{P}}=\emptyset$ it holds:

$$\begin{aligned}&b\in\pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}})\cap\pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}}) \text{ with } \pi_{\sqcup_{\check{\mathbb{P}}}}(b)=\check{e} \text{ and } \pi_{\sqcup_{\mathbb{P}}}(b)=u\\&\text{iff } b\in\{u\}\text{Ш}\{\check{e}\}.\end{aligned}\tag{248}$$

Now (248) allows to prove (247) inductively using the inductive definitions of $\text{Ш}^{(\mathbb{P})}$ and Ш.
Induction base
Let $u=\varepsilon$ or $e=\varepsilon$. We only consider $e=\varepsilon$, because $u=\varepsilon$ can be treated analogously. Then $x\in\{u\}\text{Ш}^{(\mathbb{P})}\{e\}$, iff $x=u$, iff there exists a shuffled representation $b\in\pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}})\cap\pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}})$ of x by u and $\check{e}=\varepsilon$.
Induction step
$u\neq\varepsilon\neq e$ implies $u=u'(f,a,f')$ and $e=e'(g,b,g')$ with $u'\in A_{\mathbb{P}}$, $e'\in E_{\mathbb{P}}$, and $(f,a,f'),(g,b,g')\in\sqcup_{\mathbb{P}}$. Therefore, $x\in\{u\}\text{Ш}^{(\mathbb{P})}\{e\}$ implies $x\in(\{u'\}\text{Ш}^{(\mathbb{P})}\{e'(g,b,g')\})(f+g',a,f'+g')\cup(\{u'(f,a,f')\}\text{Ш}^{(\mathbb{P})}\{e'\})(f'+g,b,f'+g')$. On account of symmetry it is sufficient to prove the induction step for $x\in(\{u'\}\text{Ш}^{(\mathbb{P})}\{e'(g,b,g')\})(f+g',a,f'+g')$, which implies $x=x'(f+g',a,f'+g')$ with $x'\in\{u'\}\text{Ш}^{(\mathbb{P})}\{e'(g,b,g')\}$. Now by the induction hypothesis $x'\in\{u'\}\text{Ш}^{(\mathbb{P})}\{e'(g,b,g')\}$ implies the existence of a shuffled representation $b'\in\pi_{\sqcup_{\mathbb{P}}}^{-1}(A_{\mathbb{P}})\cap\pi_{\sqcup_{\check{\mathbb{P}}}}^{-1}(E_{\check{\mathbb{P}}})$ of x' by u' and $\check{e}=\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(e')(g,\check{b},g')$. But then $b:=b'(f,a,f')$ is a shuffled representation of $x=x'(f+g',a,f'+g')$ by $u=u'(f,a,f')$ and $\check{e}=\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(e')(g,\check{b},g')$.
Let now b be a shuffled representation of x by $u=u'(f,a,f')$ and $\check{e}=\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(e')(g,\check{b},g')$. Then $b\in\{u\}\text{Ш}\{\check{e}\}=(\{u'\}\text{Ш}\{\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(e')(g,\check{b},g')\})(f,a,f')\cup(\{u'(f,a,f')\}\text{Ш}^{(\mathbb{P})}\{\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(e')\})(g,\check{b},g')$. On account of symmetry it is sufficient to prove the induction step for $b\in(\{u'\}\text{Ш}\{\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(e')(g,\check{b},g')\})(f,a,f')$, which implies $b=b'(f,a,f')$ with $b'\in(\{u'\}\text{Ш}\{\check{\iota}_{\sqcup_{\check{\mathbb{P}}}}^{-1}(e')(g,\check{b},g')\})$. Additionally b' is a shuffled

representation of x' by u' and $\check{e} = \check{\iota}^{-1}_{\sqcup_{\check{\mathbb{P}}}}(e')(g,\check{b},g')$ with $x = x'(f+g',a,f'+g')$. Now by the induction hypothesis $x' \in \{u'\}\text{Ш}^{(\mathbb{P})}\{e'(g,b,g')\}$, which implies $x = x'(f+g',a,f'+g') \in \{u'(f,a,f'\}\text{Ш}^{(\mathbb{P})}\{e'(g,b,g')\}$. This completes the induction step and the proof of Theorem 22.

Analogously to the proofs of Theorem 22 and Theorem 16 a representation of $\mathrm{SRF}_{\mathbb{P}}$ can be constructed like such in Theorem 16.

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