Proofs for: Construction Principles for Well-behaved Scalable Systems

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Theorem 1 (intersection theorem)**.** *Let $\mathcal{I}$ be a parameter structure, $\mathcal{B}_\mathcal{I}$ an isomorphism structure for $\mathcal{I}$, and $T \neq \emptyset$.*

i) Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ for each $t \in T$ be a monotonic parameterised system, then $(\bigcap_{t\in T} \mathcal{L}^t_I)_{I\in\mathcal{I}}$ is a monotonic parameterised system.

ii) Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ for each $t \in T$ be a scalable system with respect to $\mathcal{B}_\mathcal{I}$, then $(\bigcap_{t\in T} \mathcal{L}^t_I)_{I\in\mathcal{I}}$ is a scalable system with respect to $\mathcal{B}_\mathcal{I}$.

iii) Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ for each $t \in T$ be a self-similar monotonic parameterised system, then $(\bigcap_{t\in T} \mathcal{L}^t_I)_{I\in\mathcal{I}}$ is a self-similar monotonic parameterised system.

Proof of Theorem 1 (i)–(iii):

Proof of (i): Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ a monotonic parameterised system for each $t \in T$, then $\mathcal{L}^t_{I'} \subset \mathcal{L}^t_I$ for $t \in T$, $I, I' \in \mathcal{I}$, and $I' \subset I$. This implies

$$\bigcap_{t\in T} \mathcal{L}^t_{I'} \subset \bigcap_{t\in T} \mathcal{L}^t_I.$$

So, $(\bigcap_{t\in T} \mathcal{L}^t_I)_{I\in\mathcal{I}}$ is a monotonic parameterised system.

Proof of (ii): Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ an scalable system with respect to $(\mathcal{B}(I,K))_{(I,K)\in\mathcal{I}\times\mathcal{I}}$ for each $t \in T$, then $\iota^I_K(\mathcal{L}^t_I) = \mathcal{L}^t_K$ for $t \in T$, I, $K \in \mathcal{I}$, and $\iota \in \mathcal{B}(I,K)$.

Because all ι^I_K are isomorphisms,

$$\iota^I_K(\bigcap_{t\in T} \mathcal{L}^t_I) = \bigcap_{t\in T} \iota^I_K(\mathcal{L}^t_I) = \bigcap_{t\in T} \mathcal{L}^t_K.$$

Therefore, $(\bigcap_{t\in T} \mathcal{L}^t_I)_{I\in\mathcal{I}}$ is a scalable system with respect to $(\mathcal{B}(I,K))_{(I,K)\in\mathcal{I}\times\mathcal{I}}$.

Proof of (iii): Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ a self-similar monotonic parameterised system for each $t \in T$. For $I, I' \in \mathcal{I}$ with $I' \subset I$ holds

$$\Pi^I_{I'}(\bigcap_{t\in T} \mathcal{L}^t_I) \subset \bigcap_{t\in T} \Pi^I_{I'}(\mathcal{L}^t_I) = \bigcap_{t\in T} \mathcal{L}^t_{I'} \subset \bigcap_{t\in T} \mathcal{L}^t_I. \quad (1)$$

Because $\bigcap_{t\in T} \mathcal{L}^t_{I'} \subset \Sigma^*_{I'}$ holds

$$\Pi^I_{I'}(\bigcap_{t\in T} \mathcal{L}^t_{I'}) = \bigcap_{t\in T} \mathcal{L}^t_{I'}.$$

Together with the second inclusion from (1) it follows

$$\bigcap_{t\in T} \mathcal{L}^t_{I'} \subset \Pi^I_{I'}(\bigcap_{t\in T} \mathcal{L}^t_I).$$

Because of the first part of (1) now holds

$$\Pi^I_{I'}(\bigcap_{t\in T} \mathcal{L}^t_I) = \bigcap_{t\in T} \mathcal{L}^t_{I'}.$$

Therefore,

$$(\bigcap_{t\in T} \mathcal{L}^t_I)_{I\in\mathcal{I}}$$

is a self-similar monotonic parameterised system with respect to $\mathcal{I}$. ■

Theorem 2 (simplest well-behaved scalable systems)**.** *$(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ is a well-behaved scalable system with respect to each isomorphism structure for $\mathcal{I}$ based on N and*

$$\dot{\mathcal{L}}(L)_I = \bigcap_{i\in N} (\tau^I_i)^{-1}(L) \text{ for each } I \in \mathcal{I}.$$

The proof of Theorem 2 will be given in context of influence structures because it consists of special cases of more general results on influence structures (see 32).

Further requirements, which assure that $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ are well-behaved scalable systems, will be given with respect to $\mathcal{E}_\mathcal{I}$, $\mathcal{B}_\mathcal{I}$, L and V. This will be prepared by some lemmata.

Lemma 1. *Let $\mathcal{E}_\mathcal{I} := (E(t,I))_{(t,I)\in T\times\mathcal{I}}$ be an influence structure for $\mathcal{I}$ indexed by T, and let $V \subset \Sigma^*$. If*

$$E(t,I') = E(t,I) \cap I' \quad (2)$$

for each $t \in T$ and $I, I' \in \mathcal{I}$ $I' \subset I$, then

$$((\tau_{E(t,I)})^{-1}(V))_{I\in\mathcal{I}}$$

is a monotonic parameterised system for each $t \in T$, and by the intersection theorem

$$(\bigcap_{t\in T} (\tau_{E(t,I)})^{-1}(V))_{I\in\mathcal{I}}$$

is a monotonic parameterised system.

Proof: Let $I \in \mathcal{I}$ and $t \in T$. From the definitions of influence homomorphisms and influence structures it follows

$$\tau^I_{E(t,I)}(a_i) = \begin{cases} a \mid & a_i \in \Sigma_{E(t,I)} \\ \varepsilon \mid & a_i \in \Sigma_I \setminus \Sigma_{E(t,I)} \end{cases} . \quad (3)$$

For $I' \subset I$, $I' \in \mathcal{I}$ and $a_i \in \Sigma_{I'}$ then because of (2)

$$\begin{aligned} \tau^I_{E(t,I)}(a_i) &= \begin{cases} a \mid & a_i \in \Sigma_{E(t,I)} \cap \Sigma_{I'} \\ \varepsilon \mid & a_i \in \Sigma_{I'} \cap \Sigma_I \setminus \Sigma_{E(t,I)} \end{cases} \\ &= \begin{cases} a \mid & a_i \in \Sigma_{E(t,I')} \\ \varepsilon \mid & a_i \in \Sigma_{I'} \setminus (\Sigma_{E(t,I)} \cap \Sigma_{I'}) \end{cases} \\ &= \begin{cases} a \mid & a_i \in \Sigma_{E(t,I')} \\ \varepsilon \mid & a_i \in \Sigma_{I'} \setminus \Sigma_{E(t,I')} \end{cases} = \tau^{I'}_{E(t,I')}(a_i), \end{aligned}$$

and therefore

$$(\tau^{I'}_{E(t,I')})^{-1}(V) \subset (\tau^I_{E(t,I)})^{-1}(V) \text{ for } V \subset \Sigma^*. \quad (4)$$

So,

$$((\tau^I_{E(t,I)})^{-1}(V))_{I \in \mathcal{I}} \quad (5)$$

is a monotonic parameterised system for each $t \in T$. ■

Example 1. *Let $\mathcal{I}$ be a parameter structure based on N. For $I \in \mathcal{I}$ and $i \in N$ let:*

$$\dot{E}(i,I) := \begin{cases} \{i\} \mid & i \in I \\ \emptyset \mid & i \in N \setminus I \end{cases} .$$

By the definition of parameter structure $N \neq \emptyset$. So

$$\dot{\mathcal{E}}_{\mathcal{I}} := (\dot{E}(i,I))_{(i,I) \in N \times \mathcal{I}}$$

defines an influence structure for $\mathcal{I}$ indexed by N. $\dot{\mathcal{E}}_{\mathcal{I}}$ satisfies (2) and by $\tau^I_i = \tau^I_{\{i\}}$ $\tau^I_i = \tau^I_{\dot{E}(i,I)}$ for $i \in N$ and $I \in \mathcal{I}$.

Now by Lemma 1 for $V \subset \Sigma^*$

$$((\tau^I_i)^{-1}(V))_{I \in \mathcal{I}} \text{ is a monotonic parameterised system} \quad (6)$$

for each $i \in N$.

For this special influence structure $\dot{\mathcal{E}}_{\mathcal{I}}$ a stronger result can be obtained.

Lemma 2. *Let $\mathcal{I}$ be a parameter structure based on N and $\varepsilon \in L \subset \Sigma^*$. Then*

$$((\tau^I_i)^{-1}(L))_{I \in \mathcal{I}}$$

is a self-similar monotonic parameterised system for each $i \in N$, and by the intersection theorem

$$(\bigcap_{i \in N} (\tau^I_i)^{-1}(L))_{I \in \mathcal{I}}$$

is a self-similar monotonic parameterised system.

Proof: On account of (6)

$$\Pi^I_{I'}((\tau^I_i)^{-1}(L)) = (\tau^{I'}_i)^{-1}(L)$$

has to be shown for $I, I' \in \mathcal{I}$, $I' \subset I$, and $i \in N$.

(6) implies $(\tau^{I'}_i)^{-1}(L) \subset (\tau^I_i)^{-1}(L)$ and therefore,

$$(\tau^{I'}_i)^{-1}(L) = \Pi^I_{I'}((\tau^{I'}_i)^{-1}(L)) \subset \Pi^I_{I'}((\tau^I_i)^{-1}(L)). \quad (7)$$

It remains to show $\Pi^I_{I'}((\tau^I_i)^{-1}(L)) \subset (\tau^{I'}_i)^{-1}(L)$.

Case 1. $i \notin I'$

Because of $\varepsilon \in L$ and $\tau^{I'}_i(w) = \varepsilon$ for $i \notin I'$ and $w \in \Sigma^*_{I'}$, it holds $(\tau^{I'}_i)^{-1}(L) = \Sigma^*_{I'}$ and so

$$\Pi^I_{I'}((\tau^I_i)^{-1}(L)) \subset (\tau^{I'}_i)^{-1}(L) \text{ for } i \notin I'. \quad (8)$$

Case 2. $i \in I'$

From definitions of $\Pi^I_{I'}, \tau^I_i$ and $\tau^{I'}_i$ follows

$$\tau^I_i = \tau^{I'}_i \circ \Pi^I_{I'} \text{ for } i \in I'. \quad (9)$$

For $x \in \Pi^I_{I'}((\tau^I_i)^{-1}(L))$ exists $y \in \Sigma^*_I$ with $\tau^I_i(y) \in L$ and $x = \Pi^I_{I'}(y)$. Because of (9) holds

$$\tau^{I'}_i(x) = \tau^{I'}_i(\Pi^I_{I'}(y)) = \tau^I_i(y) \in L,$$

hence, $x \in (\tau^{I'}_i)^{-1}(L)$. Therefore,

$$\Pi^I_{I'}((\tau^I_i)^{-1}(L)) \subset (\tau^{I'}_i)^{-1}(L) \text{ for } i \in I'. \quad (10)$$

Because of (8), (10) and (7) holds

$$\Pi^I_{I'}((\tau^I_i)^{-1}(L)) = (\tau^{I'}_i)^{-1}(L)$$

for $I, I' \in \mathcal{I}$, $I' \subset I$ and $i \in N$. ■

Intersections of system behaviours play an important role concerning uniformity of parameterisation. Therefore, some general properties of intersections of families of sets will be presented.

Let T be a set. A family $f = (f_t)_{t \in T}$ with $f_t \in F$ for each $t \in T$ is formally equivalent to a function $f : T \to F$ with $f_t := f(t)$.

Let M be a set. A family $f = (f_t)_{t \in T}$ with $f_t \in F = \mathcal{P}(M)$ for each $t \in T$ is called a family of subsets of M.

Let now $T \neq \emptyset$ and f a family of subsets of M. The intersection $\bigcap_{t \in T} f_t$ is defined by

$$\bigcap_{t \in T} f_t = \{m \in M | m \in f_t \text{ for each } t \in T\}. \quad (11)$$

If $f = g \circ h$ with $h : T \to H$ and $g : H \to F$ then

$$\bigcap_{t \in T} f(t) = \bigcap_{x \in h(T)} g(x). \quad (12)$$

If especially $f = h$ and g is the identity on F, then from (12) follows

$$\bigcap_{t \in T} f(t) = \bigcap_{x \in f(T)} x.$$

For a second family of sets $f' : T' \to F$ with $f'(T') = f(T)$ follows then

$$\bigcap_{t \in T} f(t) = \bigcap_{t' \in T'} f(t').$$

In the following we will use family and function notations side by side.

Let $f=(f_t)_{t\in T}$ a family of sets with $f:T\to F=\mathcal{P}(M)$. If $T=\mathring{T}\cup\hat{T}$ with $\mathring{T}\neq\emptyset$ and $f(\hat{T})=\{M\}$, then from (11) follows

$$\bigcap_{t\in T} f(t)=\bigcap_{t\in\mathring{T}} f(t). \tag{13}$$

Let $\mathcal{E}_\mathcal{I}=(E(t,I))_{(t,I)\in T\times\mathcal{I}}$ be an influence structure for $\mathcal{I}$ indexed by T.

For each $I\in\mathcal{I}$ a family of sets

$$\mathcal{E}_\mathcal{I}(I):=(E(t,I))_{t\in T}$$

with $E(t,I)=\mathcal{E}_\mathcal{I}(I)(t)\in\mathcal{P}(I)$ is defined, and it holds

$$\mathcal{E}_\mathcal{I}(I):T\to\mathcal{P}(I). \tag{14}$$

From (12) it follows (with $h=\mathcal{E}_\mathcal{I}(I)$)

$$\bigcap_{t\in T}(\tau^I_{E(t,I)})^{-1}(V)=\bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T)}(\tau^I_x)^{-1}(V) \tag{15}$$

for each $V\subset\Sigma^*$ and $I\in\mathcal{I}$.

For each $I\in\mathcal{I}$ holds $\tau^I_\emptyset(w)=\varepsilon$ for each $w\in\Sigma^*_I$. It follows,

$$(\tau^I_\emptyset)^{-1}(V)=\Sigma^*_I \text{ if } \varepsilon\in V\subset\Sigma^*. \tag{16}$$

Because of (12), (13), (15), and (16)

$$\begin{aligned}\bigcap_{t\in T}(\tau^I_{E(t,I)})^{-1}(V)&=\bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T_I)}(\tau^I_x)^{-1}(V)\\&=\bigcap_{t\in T_I}(\tau^I_{E(t,I)})^{-1}(V)\end{aligned} \tag{17}$$

for each T_I with $\emptyset\neq T_I\subset T$ and $\mathcal{E}_\mathcal{I}(I)(T)\setminus\mathcal{E}_\mathcal{I}(I)(T_I)\in\{\emptyset,\{\emptyset\}\}$ and $\varepsilon\in V\subset\Sigma^*$.

Each bijection $\iota:I\to I'$ defines another bijection $\breve{\iota}:\mathcal{P}(I)\to\mathcal{P}(I')$ by

$$\breve{\iota}(x):=\{\iota(y)\in I'|y\in x\} \text{ for each } x\in\mathcal{P}(I). \tag{18}$$

Lemma 3. *Let $\mathcal{E}_\mathcal{I}=(E(t,I))_{(t,I)\in T\times\mathcal{I}}$ be an influence structure for $\mathcal{I}$ indexed by T, and let $\mathcal{B}_\mathcal{I}=(\mathcal{B}(I,I'))_{(I,I')\in\mathcal{I}\times\mathcal{I}}$ be an isomorphism structure for $\mathcal{I}$. Let*

$\varepsilon\in V\subset\Sigma^*$, and let $(T_K)_{K\in\mathcal{I}}$ be a family with $\emptyset\neq T_K\subset T$ and $\mathcal{E}_\mathcal{I}(K)(T)\setminus\mathcal{E}_\mathcal{I}(K)(T_K)\in\{\emptyset,\{\emptyset\}\}$ for each $K\in\mathcal{I}$, such that $\breve{\iota}(\mathcal{E}_\mathcal{I}(I)(T_I))=\mathcal{E}_\mathcal{I}(I')(T_{I'})$ for each $(I,I')\in\mathcal{I}\times\mathcal{I}$ and $\iota\in\mathcal{B}(I,I')$, (19)

then

$$\bigcap_{t\in T}(\tau^I_{E(t,I)})^{-1}(V)=\bigcap_{t\in T_I}(\tau^I_{E(t,I)})^{-1}(V) \tag{20}$$

for each $I\in\mathcal{I}$, and

$$\iota^I_{I'}[\bigcap_{t\in T}(\tau^I_{E(t,I)})^{-1}(V)]=\bigcap_{t\in T}(\tau^{I'}_{E(t,I')})^{-1}(V) \tag{21}$$

for each $(I,I')\in\mathcal{I}\times\mathcal{I}$ and $\iota\in\mathcal{B}(I,I')$.

Proof of (20): Because of (17) from assumption (19) directly follows (20). ■

For the proof of (21) the following property of the homomorphisms τ^I_K is needed:

Let $\iota:I\to I'$ a bijection and $K\subset I$, then $\tau^{I'}_{\iota(K)}\circ\iota^I_{I'}=\tau^I_K$ and so

$$\tau^{I'}_{\iota(K)}=\tau^I_K\circ(\iota^I_{I'})^{-1}. \tag{22}$$

Proof of (22):

The elements of Σ_I are of the form a_i with $i\in I$ and $a\in\Sigma$. For these elements holds

$$\begin{aligned}\tau^I_K(a_i)&=\begin{cases}a\;|&i\in K\\\varepsilon\;|&i\in I\setminus K\end{cases}\\&=\begin{cases}a\;|&\iota(i)\in\iota(K)\\\varepsilon\;|&\iota(i)\in I'\setminus\iota(K)\end{cases}\\&=\tau^{I'}_{\iota(K)}(a_{\iota(i)})=\tau^{I'}_{\iota(K)}(\iota^I_{I'}(a_i))\end{aligned}$$

which proves (22). ■

Proof of (21): Because of (17) and (22)

$$\begin{aligned}&\iota^I_{I'}[\bigcap_{t\in T}(\tau^I_{E(t,I)})^{-1}(V)]\\&=\iota^I_{I'}[\bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T_I)}(\tau^I_x)^{-1}(V)]\\&=((\iota^I_{I'})^{-1})^{-1}[\bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T_I)}(\tau^I_x)^{-1}(V)]\\&=\bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T_I)}((\iota^I_{I'})^{-1})^{-1}[(\tau^I_x)^{-1}(V)]\\&=\bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T_I)}(\tau^I_x\circ(\iota^I_{I'})^{-1})^{-1}(V)\\&=\bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T_I)}(\tau^{I'}_{\iota(x)})^{-1}(V)\\&=\bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T_I)}(\tau^{I'}_{\breve{\iota}(x)})^{-1}(V).\end{aligned} \tag{23}$$

From (12) (with $h=\breve{\iota}$) and the assumption (19) follows

$$\begin{aligned}\bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T_I)}(\tau^{I'}_{\breve{\iota}(x)})^{-1}(V)&=\bigcap_{x'\in\breve{\iota}(\mathcal{E}_\mathcal{I}(I)(T_I))}(\tau^{I'}_{x'})^{-1}(V)\\&=\bigcap_{x'\in\mathcal{E}_\mathcal{I}(I')(T'_I)}(\tau^{I'}_{x'})^{-1}(V).\end{aligned}$$

Furthermore, from (17) follows

$$\bigcap_{x'\in\mathcal{E}_\mathcal{I}(I')(T'_I)}(\tau^{I'}_{x'})^{-1}(V)=\bigcap_{t\in T}(\tau^{I'}_{E(t,I')})^{-1}(V). \tag{24}$$

(23) - (24) prove (21). ■

The case $T=N$, where $\mathcal{I}$ is based on N, allows a simpler sufficient condition for (20) and (21).

Lemma 4. *Let $\mathcal{I}$ be a parameter structure based on N, $\mathcal{E}_\mathcal{I}=(E(n,I))_{(n,I)\in N\times\mathcal{I}}$ be an influence structure for $\mathcal{I}$, and let $\mathcal{B}_\mathcal{I}=(\mathcal{B}(I,I'))_{(I,I')\in\mathcal{I}\times\mathcal{I}}$ be an isomorphism* *structure for $\mathcal{I}$.*

$$\text{Let } \varepsilon \in V \subset \Sigma^*, \tag{25a}$$

$$\text{for each } I \in \mathcal{I} \text{ and } n \in N \text{ let } E(n,I) = \emptyset, \text{ or it exists an } i_n \in I \text{ with } E(n,I) = E(i_n,I), \text{ and} \tag{25b}$$

$$\text{for each } (I,I') \in \mathcal{I}\times\mathcal{I}, \iota \in \mathcal{B}(I,I') \text{ and } i \in I \text{ holds } \iota(E(i,I)) = E(\iota(i),I'). \tag{25c}$$

Then

$$\bigcap_{n\in N} (\tau^I_{E(n,I)})^{-1}(V) = \bigcap_{n\in I} (\tau^I_{E(n,I)})^{-1}(V) \tag{26}$$

for each $I \in \mathcal{I}$, and

$$\iota^I_{I'}[\bigcap_{n\in N} (\tau^I_{E(n,I)})^{-1}(V)] = \bigcap_{n\in N} (\tau^{I'}_{E(n,I')})^{-1}(V) \tag{27}$$

for each $(I,I') \in \mathcal{I}\times\mathcal{I}$ and $\iota \in \mathcal{B}(I,I')$.

Proof: From (25b) follows $\mathcal{E}_\mathcal{I}(I)(N) = \mathcal{E}_\mathcal{I}(I)(I)$ or $\mathcal{E}_\mathcal{I}(I)(N) = \mathcal{E}_\mathcal{I}(I)(I) \cup \{\emptyset\}$, so

$$\mathcal{E}_\mathcal{I}(I)(N) \setminus \mathcal{E}_\mathcal{I}(I)(I) \in \{\emptyset, \{\emptyset\}\} \text{ for each } I \in \mathcal{I}. \tag{28}$$

From (25c) follows

$$\breve{\iota}(\mathcal{E}_\mathcal{I}(I)(I)) \subset \mathcal{E}_\mathcal{I}(I')(I'). \tag{29}$$

Because $\iota: I \to I'$ is a bijection, for each $i' \in I'$ exists an $i \in I$ with $\iota(i) = i'$. Because of (25c) holds $\breve{\iota}(E(i,I)) = E(i',I')$, where $E(i,I) \in \mathcal{E}_\mathcal{I}(I)(I)$. From this follows

$$\mathcal{E}_\mathcal{I}(I')(I') \subset \breve{\iota}(\mathcal{E}_\mathcal{I}(I)(I)). \tag{30}$$

Because of (28) - (30), with $T = N$ and $(T_I)_{I\in\mathcal{I}} = (I)_{I\in\mathcal{I}}$,

$$(25a) - (25c) \text{ implies } (19).$$

■

Example 2 (Example 1 (continued))**.** *Let $\mathcal{I}$ be a parameter structure based on N and $\mathcal{B}_\mathcal{I} = (\mathcal{B}(I,I'))_{(I,I')\in\mathcal{I}\times\mathcal{I}}$ be an isomorphism structure for $\mathcal{I}$. Then $\dot{\mathcal{E}}_\mathcal{I}$ satisfies (25b) and (25c).*

So for $\varepsilon \in L \subset \Sigma^*$ Lemma 4 implies

$$\bigcap_{n\in N} (\tau^I_n)^{-1}(L) = \bigcap_{n\in I} (\tau^I_n)^{-1}(L) \text{ for each } I \in \mathcal{I} \text{ and}$$
$$\iota^I_{I'}[\bigcap_{n\in N} (\tau^I_n)^{-1}(L)] = \bigcap_{n\in N} (\tau^{I'}_n)^{-1}(L) \tag{31}$$

for each $(I,I') \in \mathcal{I}\times\mathcal{I}$ and $\iota \in \mathcal{B}(I,I')$.

Now Lemma 2 together with (31) proves Theorem 2. (32)

Because of $\tau^I_n = \tau^I_{\dot{E}(n,I)}$ for $I \in \mathcal{I}$ and $n \in N$, (31) and the definitions of $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ and $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ imply

$$\begin{aligned}\dot{\mathcal{L}}(L)_I &= \bigcap_{n\in I} (\tau^I_n)^{-1}(L) = \bigcap_{n\in I} (\tau^I_n)^{-1}(L) \cap \bigcap_{n\in I} (\tau^I_n)^{-1}(V)\\ &= \dot{\mathcal{L}}(L)_I \cap \bigcap_{n\in N} (\tau^I_n)^{-1}(V)\\ &= \dot{\mathcal{L}}(L)_I \cap \bigcap_{n\in N} (\tau^I_{\dot{E}(n,I)})^{-1}(V)\\ &= \mathcal{L}(L,\dot{\mathcal{E}}_I,V)_I\end{aligned} \tag{33}$$

for $I \in \mathcal{I}$ and $V \supset L$.

(33) gives a representation of $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ in terms of $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$.

For the following theorems please remember that by the general definition of $\mathcal{L}(L,\mathcal{E}_I,V)_I$ it is assumed that $\emptyset \neq L \subset V$ and L,V are prefix closed. This implies $\varepsilon \in L \subset V$.

Lemma 5. *Let $\mathcal{I}$ be a parameter structure, $\mathcal{E}_\mathcal{I}$ an influence structure for $\mathcal{I}$ indexed by T and $\mathcal{B}_\mathcal{I}$ an isomorphism structure for $\mathcal{I}$.*

Assuming (2) and (19), then

$$(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$$

is a scalable systems with respect to $\mathcal{B}_\mathcal{I}$.

$$\textit{It holds } \mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I = \dot{\mathcal{L}}(L)_I \cap \bigcap_{n\in T_I} (\tau^I_{E(n,I)})^{-1}(V)$$

for each $I \in \mathcal{I}$.

Proof: By Theorem 2, $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ is a scalable system with respect to $\mathcal{B}_\mathcal{I}$. By Lemma 1 and 3 (21)

$$(\bigcap_{t\in T} (\tau^I_{E(t,I)})^{-1}(V))_{I\in\mathcal{I}}$$

is a scalable system with respect to $\mathcal{B}_\mathcal{I}$ too. Now part (ii) of the intersection theorem proves $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ to be a scalable system with respect to $\mathcal{B}_\mathcal{I}$. Lemma 3 (20) completes the proof of Lemma 5. ■

Using Lemma 4 instead of Lemma 3 proves the following.

Theorem 3 (construction condition for scalable systems)**.** *By the assumptions of Lemma 4 and (2) with $T = N$, $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ is a scalable system with respect to $\mathcal{B}_\mathcal{I}$. It holds $\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I = \dot{\mathcal{L}}(L)_I \cap \bigcap_{n\in I} (\tau^I_{E(n,I)})^{-1}(V))$.*

Remark 1. *It can be shown that in $\mathrm{SP}(L,V)$ $\mathbb{N}$ can be replaced by each countable infinite set. Let therefore N' be another set and $\iota: \mathbb{N} \to N'$ a bijection. $\iota^\mathbb{N}_{N'}: \Sigma^*_\mathbb{N} \to \Sigma^*_{N'}$ is the isomorphism defined as in the definition of isomorphism structure. It now holds*

$$\Theta^\mathbb{N} = \Theta^{N'} \circ \iota^\mathbb{N}_{N'} \textit{ and } \tau^\mathbb{N}_n = \tau^{N'}_{\iota(n)} \circ \iota^\mathbb{N}_{N'} \tag{34}$$

for each $n \in \mathbb{N}$. Furthermore,

$$\iota^\mathbb{N}_{N'} \circ \Pi^\mathbb{N}_K = \Pi^{N'}_{\iota(K)} \circ \iota^\mathbb{N}_{N'} \tag{35}$$

for each $K \subset \mathbb{N}$. From (34) and commutativity of intersection now

$$(\bigcap_{n\in\mathbb{N}}(\tau_n^{\mathbb{N}})^{-1}(L))\cap(\Theta^{\mathbb{N}})^{-1}(V) = $$
$$= (\iota_{N'}^{\mathbb{N}})^{-1}[(\bigcap_{n\in\mathbb{N}}(\tau_{\iota(n)}^{N'})^{-1}(L))\cap(\Theta^{N'})^{-1}(V)]$$
$$= (\iota_{N'}^{\mathbb{N}})^{-1}[(\bigcap_{n'\in N'}(\tau_{n'}^{N'})^{-1}(L))\cap(\Theta^{N'})^{-1}(V)]. \quad (36)$$

By (35),

$$\Pi_K^{\mathbb{N}}\circ(\iota_{N'}^{\mathbb{N}})^{-1} = (\iota_{N'}^{\mathbb{N}})^{-1}\circ\Pi_{\iota(K)}^{N'}. \quad (37)$$

Because of (36) and (37)

$$\Pi_K^{\mathbb{N}}[(\bigcap_{n\in\mathbb{N}}(\tau_n^{\mathbb{N}})^{-1}(L))\cap(\Theta^{\mathbb{N}})^{-1}(V)] = $$
$$= (\iota_{N'}^{\mathbb{N}})^{-1}(\Pi_{\iota(K)}^{N'}[(\bigcap_{n'\in N'}(\tau_{n'}^{N'})^{-1}(L))\cap(\Theta^{N'})^{-1}(V)]). \quad (38)$$

From

$$\Pi_K^{\mathbb{N}}[(\bigcap_{n\in\mathbb{N}}(\tau_n^{\mathbb{N}})^{-1}(L))\cap(\Theta^{\mathbb{N}})^{-1}(V)]\subset(\Theta^{\mathbb{N}})^{-1}(V)$$

now follows

$$\Pi_{\iota(K)}^{N'}[(\bigcap_{n'\in N'}(\tau_{n'}^{N'})^{-1}(L))\cap(\Theta^{N'})^{-1}(V)] \subset \iota_{N'}^{\mathbb{N}}((\Theta^{\mathbb{N}})^{-1}(V)). \quad (39)$$

Because of (34) $\Theta^{\mathbb{N}}\circ(\iota_{N'}^{\mathbb{N}})^{-1} = \Theta^{N'}$ and so

$$(\Theta^{N'})^{-1}(V) = \iota_{N'}^{\mathbb{N}}((\Theta^{\mathbb{N}})^{-1}(V)).$$

Therefore, from (39) follows

$$\Pi_{\iota(K)}^{N'}[(\bigcap_{n'\in N'}(\tau_{n'}^{N'})^{-1}(L))\cap(\Theta^{N'})^{-1}(V)]\subset(\Theta^{N'})^{-1}(V). \quad (40)$$

Because for each $\emptyset\neq K'\subset N'$ it exists an $\emptyset\neq K\subset\mathbb{N}$ with $K'=\iota(K)$, by $\mathrm{SP}(L,V)$, we get for each $\emptyset\neq K\subset\mathbb{N}$ a corresponding inclusion with N' replacing $\mathbb{N}$ and K' for K.

Lemma 6. *The assumptions of Lemma 1 and Lemma 2 together with $\mathrm{SP}(L,V)$ imply that $(X(L,V,t)_I)_{I\in\mathcal{I}}$ with*

$$X(L,V,t)_I := \bigcap_{n\in N}(\tau_n^I)^{-1}(L)\cap(\tau_{E(t,I)}^I)^{-1}(V)$$

is a self-similar monotonic parameterised system for each $t\in T$.

Proof: By Lemma 1 and Lemma 2, $((\tau_{E(t,I)}^I)^{-1}(V))_{I\in\mathcal{I}}$ and $(\bigcap_{n\in N}(\tau_n^I)^{-1}(L))_{I\in\mathcal{I}}$ are monotonic parameterised systems. So by the intersection theorem $(X(L,V,t)_I)_{I\in\mathcal{I}}$ is a monotonic parameterised system for each $t\in T$. Therefore

$$X(L,V,t)_{I'} = \Pi_{I'}^I(X(L,V,t)_{I'})\subset\Pi_{I'}^I(X(L,V,t)_I)$$

for each $I,I'\in\mathcal{I}$ with $I'\subset I$. So the proof of self-similarity can be reduced to the proof of

$$\Pi_{I'}^I(X(L,V,t)_I)\subset X(L,V,t)_{I'} \quad (41)$$

for each $t\in T$ and $I,I'\in\mathcal{I}$ with $I'\subset I$.

Because by Lemma 2

$$(\bigcap_{n\in N}(\tau_n^I)^{-1}(L))_{I\in\mathcal{I}}$$

is self-similar, it holds

$$\Pi_{I'}^I(X(L,V,t)_I)\subset\Pi_{I'}^I(\bigcap_{n\in N}(\tau_n^I)^{-1}(L)) = \bigcap_{n\in N}(\tau_n^I)^{-1}(L).$$

So the proof of (41) can be reduced to the proof of

$$\Pi_{I'}^I[\bigcap_{n\in N}(\tau_n^I)^{-1}(L)\cap(\tau_{E(t,I)}^I)^{-1}(V)]\subset(\tau_{E(t,I')}^{I'})^{-1}(V) \quad (42)$$

for each $t\in T$ and $I,I'\in\mathcal{I}$ with $I'\subset I$.

For each $w\in(\bigcap_{n\in N}(\tau_n^I)^{-1}(L))\cap(\tau_{E(t,I)}^I)^{-1}(V)$ exists a $r\in\mathbb{N}$ and $u_i\in\Sigma_{E(t,I)}^*$ for $1\leq i\leq r$ and $v_i\in\Sigma_{I\setminus E(t,I)}^*$ for $1\leq i\leq r$ with $w=u_1v_1u_2v_2\dots u_rv_r$.
Note that $\Sigma_\emptyset := \emptyset$ and $\emptyset^* = \{\varepsilon\}$.
Because $u_1u_2\dots u_r\in\Sigma_{E(t,I)}^*$ and $v_1v_2\dots v_r\in\Sigma_{I\setminus E(t,I)}^*$ holds

$$\Theta^N(u_1u_2\dots u_r) = \tau_{E(t,I)}^I(u_1u_2\dots u_r)$$
$$= \tau_{E(t,I)}^I(w)\in V. \quad (43)$$

With the same argumentation holds

$$\tau_n^N(u_1u_2\dots u_r) = \tau_n^I(u_1u_2\dots u_r) = \tau_n^I(w)\in L \quad (44)$$

for $n\in E(t,I)$ and

$$\tau_n^N(u_1u_2\dots u_r) = \varepsilon\in L \quad (45)$$

for $n\in N\setminus E(t,I)$. With (43) - (45) now

$$u_1u_2\dots u_r\in(\bigcap_{n\in N}(\tau_n^N)^{-1}(L))\cap(\Theta^N)^{-1}(V),$$

and on behalf of precondition $\mathrm{SP}(L,V)$ holds

$$\Pi_{I'}^N(u_1u_2\dots u_r) = \Pi_{I'\cap E(t,I)}^{E(t,I)}(u_1u_2\dots u_r)$$
$$\in\Sigma_{I'\cap E(t,I)}^*\cap(\Theta^N)^{-1}(V). \quad (46)$$

Furthermore,

$$\Pi_{I'}^I(w) = \Pi_{I'}^I(u_1v_1u_2v_2\dots u_rv_r)$$
$$= \Pi_{I'\cap E(t,I)}^{E(t,I)}(u_1)\Pi_{I'\setminus E(t,I)}^{I\setminus E(t,I)}(v_1)\dots$$
$$\Pi_{I'\cap E(t,I)}^{E(t,I)}(u_r)\Pi_{I'\setminus E(t,I)}^{I\setminus E(t,I)}(v_r). \quad (47)$$

Because of (2), $E(t,I')\subset E(t,I)$ and so $I'\setminus E(t,I)\subset I'\setminus E(t,I')$ and thus

$$\tau_{E(t,I')}^{I'}(\Pi_{I'\setminus E(t,I)}^{I\setminus E(t,I)})(v_i) = \varepsilon$$

for $1\leq i\leq r$. With (2) and (47) it follows

$$\tau_{E(t,I')}^{I'}(\Pi_{I'}^I(w)) = \tau_{E(t,I')}^{I'}(\Pi_{E(t,I')}^{E(t,I)}(u_1\dots u_r)). \quad (48)$$

Because $\tau^{I'}_{E(t,I')}(x) = \Theta^N(x)$ for each $x \in \Sigma^*_{E(t,I')}$ now on behalf of (48), (2), and (46)

$$\tau^{I'}_{E(t,I')}(\Pi^I_{I'}(w)) = \Theta^N(\Pi^{E(t,I)}_{E(t,I')}(u_1 \ldots u_r)) \in V,$$

and thus $\Pi^I_{I'}(w) \in (\tau^{I'}_{E(t,I')})^{-1}(V)$. This proves (42) and completes the proof of Lemma 6. ∎

Because of the idempotence of intersection

$$\bigcap_{n\in N} (\tau^I_n)^{-1}(L) \cap \bigcap_{t\in T} (\tau^I_{E(t,I)})^{-1}(V) = \bigcap_{t\in T}[\bigcap_{n\in N} (\tau^I_n)^{-1}(L) \cap (\tau^I_{E(t,I)})^{-1}(V)].$$

Now the intersection theorem and Lemma 6 imply

Lemma 7. *If* SP(L,V)*, then by the assumptions of Lemma 1 and 2*

$$[\bigcap_{n\in N} (\tau^I_n)^{-1}(L) \cap \bigcap_{t\in T} (\tau^I_{E(t,I)})^{-1}(V)]_{I\in\mathcal{I}}$$

is a self-similar monotonic parameterised system.

Combining Lemma 7 with Lemma 5 or Theorem 3 imply

Theorem 4 (construction condition for well-behaved scalable systems)**.** *By the assumptions of Lemma 5 or Theorem 3 together with* SP(L,V)

$$(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$$

is a well-behaved scalable system.

Theorem 5 (inverse abstraction theorem)**.** *Let $\varphi : \Sigma^* \to \Phi^*$ be an alphabetic homomorphism and $W, X \subset \Phi^*$, then*

$$\text{SP}(W,X) \textit{ implies } \text{SP}(\varphi^{-1}(W), \varphi^{-1}(X)).$$

Proof of Theorem 5:

Let K be a non-empty set. Each alphabetic homomorphism $\varphi : \Sigma^* \to \Phi^*$ defines a homomorphism $\varphi^K : \Sigma^*_K \to \Phi^*_K$ by

$$\varphi^K(a_n) := (\varphi(a))_n \text{ for } a_n \in \Sigma_K, \text{ where } (\varepsilon)_n = \varepsilon. \quad (49)$$

If $\bar{\tau}^K_n : \Phi^*_K \to \Phi$ and $\bar{\Theta}^K : \Phi^*_K \to \Phi$ are defined analogous to τ^K_n and Θ^K, then

$$\varphi \circ \tau^K_n = \bar{\tau}^K_n \circ \varphi^K, \text{ and } \varphi \circ \Theta^K = \bar{\Theta}^K \circ \varphi^K. \quad (50)$$

Let now N be an infinite countable set. Because of (50), for $W, X \subset \Phi^*$

$$(\bigcap_{n\in N} (\tau^N_n)^{-1}(\varphi^{-1}(W))) \cap (\Theta^N)^{-1}(\varphi^{-1}(X)) = (\varphi^N)^{-1}[(\bigcap_{n\in N} (\bar{\tau}^N_n)^{-1}(W)) \cap (\bar{\Theta}^N)^{-1}(X)]. \quad (51)$$

Because of $\varphi^K(w) = \varphi^N(w)$ for $w \in \Sigma^*_K \subset \Sigma^*_N$ and $\emptyset \neq K \subset N$

$$(\varphi^K)^{-1}(Z) \subset (\varphi^N)^{-1}(Z) \text{ for } Z \subset \Phi^*_K. \quad (52)$$

If now SP(W,X), and

$$\Pi^N_K[(\varphi^N)^{-1}(Y)] = (\varphi^K)^{-1}(\bar{\Pi}^N_K[Y]) \quad (53)$$

for $Y \subset \Phi^*_N$ and $\emptyset \neq K \subset N$, where $\bar{\Pi}^N_K : \Phi^*_N \to \Phi^*_K$ is defined analogous to Π^N_K, then follows (with (50) - (53))

$$\begin{aligned}
&\Pi^N_K[(\bigcap_{n\in N} (\tau^N_n)^{-1}(\varphi^{-1}(W))) \cap (\Theta^N)^{-1}(\varphi^{-1}(X))] \\
&= (\varphi^K)^{-1}(\bar{\Pi}^N_K[(\bigcap_{n\in N} (\bar{\tau}^N_n)^{-1}(W)) \cap (\bar{\Theta}^N)^{-1}(X)]) \\
&\subset (\varphi^K)^{-1}((\bar{\Theta}^N)^{-1}(X)) \subset (\varphi^N)^{-1}((\bar{\Theta}^N)^{-1}(X)) \\
&= (\Theta^N)^{-1}(\varphi^{-1}(X)). \quad (54)
\end{aligned}$$

With (54)

$$\text{SP}(\varphi^{-1}(W), \varphi^{-1}(X)) \text{ follows from } \text{SP}(W,X), \quad (55)$$

if (53) holds.

It remains to show (53). For the proof of (53) it is sufficient to prove

$$\Pi^N_K((\varphi^N)^{-1}(y) = (\varphi^K)^{-1}(\bar{\Pi}^N_K(y)) \quad (56)$$

for each $y \in \Phi^*_N$, because of

$$\Pi^N_K((\varphi^N)^{-1}(Y) = \bigcup_{y\in Y} \Pi^N_K((\varphi^N)^{-1}(y))$$

and

$$(\varphi^K)^{-1}(\bar{\Pi}^N_K(Y)) = \bigcup_{y\in Y} (\varphi^K)^{-1}(\bar{\Pi}^N_K(y)).$$

Here, for $f : A \to B$ and $b \in B$ we use the convention

$$f^{-1}(b) = f^{-1}(\{b\}).$$

With $Y = \{y\}$ (56) is also necessary for (53), and so it is equivalent to (53).

Definition 1 ((general) projection)**.** *For arbitrary alphabets Δ and Δ' with $\Delta' \subset \Delta$* general projections $\pi^\Delta_{\Delta'} : \Delta^* \to \Delta'^*$ *are defined by*

$$\pi^\Delta_{\Delta'}(a) := \begin{cases} a \mid & a \in \Delta' \\ \varepsilon \mid & a \in \Delta \setminus \Delta' \end{cases}. \quad (57)$$

In this terminology the projections

$$\Pi^N_K : \Sigma^*_N \to \Sigma^*_K \text{ and } \bar{\Pi}^N_K : \Phi^*_N \to \Phi^*_K$$

considered until now are special cases, which we call *parameter-projections*. It holds

$$\Pi^N_K = \pi^{\Sigma_N}_{\Sigma_K} \text{ and } \bar{\Pi}^N_K = \pi^{\Phi_N}_{\Phi_K}. \quad (58)$$

Because of the different notations, in general we just use the term *projection* for both cases.

We now consider the equation (56) for the special case, where $\varphi : \Sigma^* \to \Phi^*$ is a projection, that is $\varphi = \pi^\Sigma_\Phi$ with $\Phi \subset \Sigma$. In this case also $\varphi^N : \Sigma^*_N \to \Phi^*_N$ is a projection, with

$$\varphi^N = \pi^{\Sigma_N}_{\Phi_N}. \quad (59)$$

Lemma 8 (projection-lemma)**.**
Let Δ be an alphabet, $\Delta' \subset \Delta$, $\Gamma \subset \Delta$ and $\Gamma' = \Delta' \cap \Gamma$, then

$$\pi^\Delta_{\Delta'}((\pi^\Delta_\Gamma)^{-1}(y)) = (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^\Delta_{\Delta'}(y))$$

for each $y \in \Gamma^*$.

Proof: Let $y \in \Gamma^*$. We show

$$\pi^{\Delta'}_{\Gamma'}(\pi^{\Delta}_{\Delta'}(z)) = \pi^{\Delta}_{\Delta'}(y) \text{ for each } z \in (\pi^{\Delta}_{\Gamma})^{-1}(y) \tag{60}$$

and we show that

$$\begin{aligned}&\text{for each } u \in (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y)) \text{ there exists a}\\ &v \in (\pi^{\Delta}_{\Gamma})^{-1}(y) \text{ such that } \pi^{\Delta}_{\Delta'}(v) = u.\end{aligned} \tag{61}$$

From (60) it follows that

$$\pi^{\Delta}_{\Delta'}((\pi^{\Delta}_{\Gamma})^{-1}(y)) \subset (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y))$$

and from (61) it follows that

$$(\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y)) \subset \pi^{\Delta}_{\Delta'}((\pi^{\Delta}_{\Gamma})^{-1}(y)),$$

which in turn proves Lemma 8.

Proof of (60): By definition of π^{Δ}_{Γ}, $\pi^{\Delta'}_{\Gamma'}$ and $\pi^{\Delta}_{\Delta'}$ follows

$$\pi^{\Delta'}_{\Gamma'}(\pi^{\Delta}_{\Delta'}(z)) = \pi^{\Delta}_{\Delta'}(\pi^{\Delta}_{\Gamma}(z))$$

for each $z \in \Delta^*$ and therewith (60).

Proof of (61) by induction on $y \in \Gamma^*$:
Induction base. Let $y = \varepsilon$, then $u \in (\Delta' \setminus \Gamma')^*$ for each $u \in (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y))$. From this follows

$$\pi^{\Delta}_{\Delta'}(v) = u \text{ with } v := u \in (\pi^{\Delta}_{\Gamma})^{-1}(\varepsilon).$$

Induction step. Let $y = \mathring{y}\hat{y}$ with $\mathring{y} \in \Gamma^*$ and $\hat{y} \in \Gamma$.
Case 1: $\hat{y} \in \Gamma \setminus \Gamma' = \Gamma \cap (\Delta \setminus \Delta')$
Then

$$(\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y)) = (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(\mathring{y})).$$

By induction hypothesis then for each $u \in (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y))$ it exists $\mathring{v} \in (\pi^{\Delta}_{\Gamma})^{-1}(\mathring{y})$ such that $\pi^{\Delta}_{\Delta'}(\mathring{v}) = u$.
With $v := \mathring{v}\hat{y}$ holds $\pi^{\Delta}_{\Gamma}(\mathring{v}\hat{y}) = \mathring{y}\hat{y} = y$ and hence

$$v \in (\pi^{\Delta}_{\Gamma})^{-1}(y) \text{ and } \pi^{\Delta}_{\Delta'}(v) = \pi^{\Delta}_{\Delta'}(\mathring{v}) = u.$$

Case 2: $\hat{y} \in \Gamma' \subset \Delta'$
Then $\pi^{\Delta}_{\Delta'}(y) = \pi^{\Delta}_{\Delta'}(\mathring{y})\hat{y}$. Therefore, each $u \in (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y))$ can be departed into $u = \mathring{u}\hat{y}\hat{u}$ with $\mathring{u} \in (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(\mathring{y}))$ and $\hat{u} \in (\Delta' \setminus \Gamma')^*$.
By induction hypothesis then exists $\mathring{v} \in (\pi^{\Delta}_{\Gamma})^{-1}(\mathring{y})$ such that $\pi^{\Delta}_{\Delta'}(\mathring{v}) = \mathring{u}$.
With $v := \mathring{v}\hat{y}\hat{u}$ holds $\pi^{\Delta}_{\Gamma}(\mathring{v}\hat{y}\hat{u}) = \mathring{y}\hat{y} = y$ and hence

$$v \in (\pi^{\Delta}_{\Gamma})^{-1}(y) \text{ and } \pi^{\Delta}_{\Delta'}(v) = \pi^{\Delta}_{\Delta'}(\mathring{v})\hat{y}\hat{u} = \mathring{u}\hat{y}\hat{u} = u.$$

This completes the proof of (61). ∎

For $y \in \Gamma^*$ holds

$$\pi^{\Delta}_{\Delta'}(y) = \pi^{\Gamma}_{\Delta' \cap \Gamma}(y) = \pi^{\Gamma}_{\Gamma'}(y).$$

Therewith, from Lemma 8 follows

$$\pi^{\Delta}_{\Delta'}((\pi^{\Delta}_{\Gamma})^{-1}(y)) = (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Gamma}_{\Gamma'}(y)) \text{ for each } y \in \Gamma^*. \tag{62}$$

For $\emptyset \neq K \subset N, \Phi \subset \Sigma, \Delta := \Sigma_N, \Delta' := \Sigma_K$, and $\Gamma := \Phi_N$ holds $\Gamma' = \Delta' \cap \Gamma = \Phi_K$.

Assuming $\varphi = \pi^{\Sigma}_{\Phi}$, which implies $\varphi^K = \pi^{\Sigma_K}_{\Phi_K}$, then from (62) (with (58) and (59)), follows

$$\Pi^N_K((\varphi^N)^{-1}(y)) = (\varphi^K)^{-1}(\bar{\Pi}^N_K(y))$$

for $y \in \Phi^*_N$, and so (56). With this,

$$\text{premise (53) is fulfilled for (55), when } \varphi \text{ is a projection,} \tag{63}$$

which proves Theorem 5 for projections.

Definition 2 (strict alphabetic homomorphism)**.** *Let* Σ, Φ *alphabets, and* $\varphi : \Sigma^* \to \Phi^*$ *a homomorphism. Then* φ *is called* alphabetic*, if* $\varphi(\Sigma) \subset \Phi \cup \{\varepsilon\}$*, and* φ *is called* strict alphabetic*, if* $\varphi(\Sigma) \subset \Phi$.

Each alphabetic homomorphism $\varphi : \Sigma^* \to \Phi^*$ is the composition of a projection with a strict alphabetic homomorphism, more precisely,

$$\varphi = \varphi_S \circ \pi^{\Sigma}_{\varphi^{-1}(\Phi)\cap\Sigma}, \tag{64}$$

where $\varphi_S : (\varphi^{-1}(\Phi) \cap \Sigma)^* \to \Phi^*$ is the strict alphabetic homomorphism defined by

$$\varphi_S(a) := \varphi(a) \text{ for } a \in \varphi^{-1}(\Phi) \cap \Sigma.$$

For $W, X \subset \Phi^*$ and $\varphi : \Sigma^* \to \Phi^*$ alphabetic (64) implies

$$\begin{aligned}\varphi^{-1}(W) &= (\pi^{\Sigma}_{\varphi^{-1}(\Phi)\cap\Sigma})^{-1}((\varphi_S)^{-1}(W)) \text{ and}\\ \varphi^{-1}(X) &= (\pi^{\Sigma}_{\varphi^{-1}(\Phi)\cap\Sigma})^{-1}((\varphi_S)^{-1}(X)).\end{aligned} \tag{65}$$

Now with (63) and (65) it remains to prove Theorem 5 for strict alphabetic homomorphisms. This will be done by Lemma 9, which proves (56) for strict alphabetic homomorphisms.

Lemma 9. *Let* $\varphi : \Sigma^* \to \Phi^*$ *be a strict alphabetic homomorphism, then for all* $y \in \Phi^*_N$ *and* $\emptyset \neq K \subset N$ *holds*

$$\Pi^N_K((\varphi^N)^{-1}(y)) = (\varphi^K)^{-1}(\bar{\Pi}^N_K(y)).$$

Proof: Proof by induction on y.
Induction basis: $y = \varepsilon$
Because φ^N is strict alphabetic

$$(\varphi^N)^{-1}(\varepsilon) = \{\varepsilon\} \text{ and so } \Pi^N_K((\varphi^N)^{-1}(\varepsilon)) = \{\varepsilon\}.$$

For the same reason

$$(\varphi^K)^{-1}(\bar{\Pi}^N_K(\varepsilon)) = (\varphi^K)^{-1}(\varepsilon) = \{\varepsilon\}.$$

Induction step: Let $y = y'a_t$ with $a_t \in \Phi_N$, where $a \in \Phi$ and $t \in N$. Because φ^N is alphabetic, it holds

$$(\varphi^N)^{-1}(y'a_t) = ((\varphi^N)^{-1}(y'))((\varphi^N)^{-1}(a_t)),$$

and so

$$\Pi^N_K((\varphi^N)^{-1}(y'a_t)) = \Pi^N_K((\varphi^N)^{-1}(y'))\Pi^N_K((\varphi^N)^{-1}(a_t)).$$

Also holds

$$(\varphi^K)^{-1}(\bar{\Pi}^N_K(y'a_t)) = (\varphi^K)^{-1}(\bar{\Pi}^N_K(y'))(\varphi^K)^{-1}(\bar{\Pi}^N_K(a_t)).$$

According to the induction hypothesis, it holds

$$\Pi_K^N((\varphi^N)^{-1}(y')) = (\varphi^K)^{-1}(\bar{\Pi}_K^N(y')).$$

Therefore, it remains to show

$$\Pi_K^N((\varphi^N)^{-1}(a_t)) = (\varphi^K)^{-1}(\bar{\Pi}_K^N(a_t)).$$

Case 1: $t \notin K$
Because φ^N is strict alphabetic, it holds $(\varphi^N)^{-1}(a_t) \subset \Sigma_t$, so

$$\Pi_K^N((\varphi^N)^{-1}(a_t)) = \{\varepsilon\}.$$

Additionally holds $\bar{\Pi}_K^N(a_t) = \varepsilon$, and therewith

$$(\varphi^K)^{-1}(\bar{\Pi}_K^N(a_t)) = \{\varepsilon\},$$

because φ^K is strict alphabetic.
Case 2: $t \in K$
Because φ^N is strict alphabetic, it holds

$$(\varphi^N)^{-1}(a_t) = \{b_t \in \Sigma_t | \varphi(b) = a\},$$

and therewith

$$\Pi_K^N((\varphi^N)^{-1}(a_t)) = \{b_t \in \Sigma_t | \varphi(b) = a\}.$$

$\bar{\Pi}_K^N(a_t) = a_t$ and therewith

$$(\varphi^K)^{-1}(\bar{\Pi}_K^N(a_t)) = \{b_t \in \Sigma_t | \varphi(b) = a\},$$

because φ^K is strict alphabetic. This completes the proof of Lemma 9. ■

This completes the proof of Theorem 5. ■